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The Fundamental Group of the General Linear Group

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INTRODUCTION

Let X be a topological space with base point $*$. A path in X is a continuous function f from the unit interval $[0, 1] = I$ to X with $f(0) = *$. The set of paths is equipped with the compact-open topology and forms a topological space EX with the constant path $f \equiv *$ as base point. The mapping $f \mapsto f(1)$ which projects every path onto its end point determines a continuous map $\epsilon: EX \to X$ of topological spaces with a base point. There is also a continuous mapping $\mu: EX \to E^2X$ given by $\mu(f)(x, y) = f(xy)$, for $(x, y) \in I^2$. Here we have tacitly identified the paths in the path space EX with the continuous mapping of the unit square I^2 to X. All these constructions are functorial in X and (E, ϵ, μ) possess the properties of a cotriple in the category Top_* of topological spaces with a base point. It is well known that a good deal of homotopy theory can be developed within this framework.

In order to introduce our subject, let k be a commutative unital ring. The category of augmented k-algebras behaves like a counterpart of Top_* in algebra, but we work with an equivalent category Alg_k which is that of all, not necessarily unital, algebras over k; see Section 2. Suppose given a cotriple (E, ϵ, μ) in Alg_k and a group valued functor $F: \mathrm{Alg}_k \to \mathrm{Gr}$. In joint work with Villamayor [21, Sections 14 and 15] we thought of the functor $FE: \mathrm{Alg}_k \to \mathrm{Gr}$ as the path space of F and $F\epsilon: FE \to F$ as the operation of projecting a path onto its end point. Then $F^0 = \mathrm{Im}\, F\epsilon$ becomes the connected component of F and the fiber $\mathrm{Ker}\, F\epsilon$ the loop space on F. This enabled us to define lifting of paths and define coverings $\lambda: H \to F$ of F, which in fact cover its connected component. We showed the existence of a universal covering $\Lambda: \hat{F} \to F$ which covers all the others. This is determined up to isomorphism over F. Taking our cue from topological groups, we then defined the fundamental group $\pi_1 F$ and the homogeneous space of components $\pi_0 F$ by the exact sequence of functors

$$1 \longrightarrow \pi_1 F \longrightarrow \hat{F} \xrightarrow{\Lambda} F \longrightarrow \pi_0 F \longrightarrow 1.$$

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The universal covering is characterized by Λ being a covering of F and the conditions $\pi_0\hat{F} = \pi_1\hat{F} = 1$, i.e., $\hat{F}$ is simply connected. Moreover, $\pi_1 F = \pi_1 F^0$, i.e., the fundamental group of F is, almost by definition, the fundamental group of its connected component.

On the other hand, consider an abstract group G and central extensions $\psi\colon U \to G$, i.e., with $\operatorname{Ker}\psi$ contained in the center of U. If G is perfect, i.e., $G = [G, G]$ or $H_1(G, \mathbf{Z}) = 0$, there exists a universal central extension $\Psi\colon \tilde{G} \to G$ determined up to isomorphism over G. This is characterized by Ψ being a central extension and $\tilde{G}$ being perfect and centrally closed, i.e., $H_1(\tilde{G}, \mathbf{Z}) = H_2(\tilde{G}, \mathbf{Z}) = 0$. Moreover, $\operatorname{Ker}\Psi = H_2(G, \mathbf{Z})$, the Schur multiplier of G [11; 13, Chapter 5].

A formal resemblance between the two situations emerges, which is captured in the following paradigm of corresponding notions:

group valued functor on Alg_k	group
connected	perfect
covering	central extension
simply connected	centrally closed
universal covering	universal central extension
fundamental group	Schur multiplier
$\pi_i \quad i = 0, 1$	$H_{i+1} \quad i = 0, 1.$

In [21] we took $k = \mathbf{Z}$, hence Alg_k the category of all rings, and (E, ϵ, μ) the free cotriple. If $F = \mathrm{GL}$, the stable general linear group with domain extended to all rings, then its connected component is EL, the stable elementary group, and $\mathrm{EL}(R)$ is perfect for every unital ring. Furthermore $\mathrm{GL}(R)$ is centrally closed and is the universal central extension of $\mathrm{EL}(R)$, so that $\pi_1\mathrm{GL}(R) \simeq H_2(\mathrm{EL}(R), \mathbf{Z})$, [21, Theorem 15.6]. This is a result in algebraic K-theory, since it says that $\pi_0\mathrm{GL}$ is Bass' K_1 and $\pi_1\mathrm{GL}$ is Milnor's K_2. In its proof the behavior of the general linear group GL on a free algebra, mainly brought to light by Gersten, and Quillen's localization sequence in K-theory are employed.

In this paper I examine what remains of the correspondence in case $F = \mathrm{GL}_n$, the general linear group of size n. We cannot expect such a functorial result, since there is no localization sequence; hence, the ultimate Theorem 22 is less impressive. For several reasons one needs to restrict oneself to a field k, and (E, ϵ, μ) the free cotriple in Alg_k. On the one hand, we must recast the classical work of Steinberg [19, Section 1] for algebras rather than just fields. On the other hand, we need the results of Cohn [3] and Silvester [14] on presentations of certain linear groups over free algebras; see Theorems 11 and 12. Their work can also be regarded as nonstable K-theory, as practiced by Dennis and Stein [6–9; 16–18]. Indeed the connection is given by Theorem 5 of Dennis, a proof of which he kindly allows me to publish. Here and elsewhere I prefer to present a coherent account rather than glean disparate results from the literature. Although there is overlap with the papers of Stein [17; 18], those deal exclusively with commutative rings. Besides, sticking to algebras over fields does yield sharper results, cf. Theorem 8, Proposition 18, and, as a curious bonus, Example 4. I have also included a generalization, Theorem 9, of a result I learn from A. Bak, partly because it has interesting consequences (see Theorem 10), partly because his telling me about it is what originally gave me the impetus toward this entire investigation.

1. Universal Central Extensions

If R is a unital ring, one defines for $n \geqslant 2$ the Steinberg group $\mathrm{ST}(n, R)$ as the group generated by all elements $x_{ij}(r)$, $1 \leqslant i \neq j \leqslant n$, r ranging over R, subject to the relations

$$\begin{array}{llll}
\text{(i)} \quad x_{ij}(r)\, x_{ij}(s) & = x_{ij}(r+s) & & r, s \in R; \\
\text{(ii)} \quad [x_{ij}(r), x_{kl}(s)] & = 1 & \text{if } i \neq l, j \neq k, & r, s \in R; \\
 & = x_{il}(rs) & \text{if } i \neq l, j = k; & \\
\text{(iii)} \quad {}^{w_{ij}(u)}x_{ji}(r) & = x_{ij}(-uru) & & r \in R, u \in R^{\bullet}
\end{array}$$

and their consequences. Here $[\alpha, \beta]$ stands for the commutator $\alpha\beta\alpha^{-1}\beta^{-1}$, ${}^{\alpha}\beta$ for the conjugate $\alpha\beta\alpha^{-1}$, and $R^{\bullet}$ for the group of (two-sided) units of R; the element $w_{ij}(u)$ is defined as $x_{ij}(u)\, x_{ji}(-u^{-1})\, x_{ij}(u)$. It is well known that relation (ii) is vacuous for $n = 2$, while for $n \geqslant 3$, (iii) follows from the first two.

Write $e_{ij}(r)$ for the n-square matrix which is the sum of the identity matrix and a matrix whose only nonnull entry is r at the intersection of the ith row and jth column. Mapping $x_{ij}(r)$ to the elementary matrix $e_{ij}(r)$ yields a homomorphism ρ of $\mathrm{ST}(n, R)$ onto $\mathrm{EL}(n, R)$. The significance of this construction lies in Steinberg's celebrated result that $\rho\colon \mathrm{ST}(n, R) \to \mathrm{EL}(n, R)$ is the universal central extension of the elementary group when R is a field, with the exception of a few very small fields.

From the standpoint of central group extensions [11; 13, Chapter 5], this is tantamount to proving that

(I) the extension ρ is central, i.e., $\operatorname{Ker} \rho$ is contained in the center of $\mathrm{ST}(n, R)$;

(II) the group $\mathrm{ST}(n, R)$ is perfect (equals its own commutator subgroup), i.e., the first homology group $H_1(\mathrm{ST}(n, R), \mathbf{Z}) = 0$;

(III) the group $\mathrm{ST}(n, R)$ is centrally closed, i.e., every central extension of $\mathrm{ST}(n, R)$ splits, or $H_2(\mathrm{ST}(n, R), \mathbf{Z}) = 0$.

Here the action of $\mathrm{ST}(n, R)$ on $\mathbf{Z}$ is understood to be the trivial one.

The group Ker ρ is denoted by $K_2(n, R)$ since it is a nonstable counterpart of Milnor's $K_2(R)$. If the three conditions are satisfied for a ring R, this means that $\mathrm{ST}(n, R)$ is the universal central extension of $\mathrm{EL}(n, R)$ (determined up to isomorphism) and $K_2(n, R) = H_2(\mathrm{EL}(n, R), \mathbf{Z})$ is its Schur multiplier. It was already noted by Stein [17, Theorem 5.3, 4] that a lot remains true for commutative rings with sufficiently many units. Deferring the discussion of (I) until Section 5, we examine in detail the last two conditions for unital algebras A over a (commutative) field k. This means k is in the center of the (not necessarily commutative) algebra A.

We first recall some facts about Steinberg groups which we need in this and other sections. For $u \in R^{\bullet}$, R any unital ring, write $h_{ij}(u) = w_{ij}(u)\, w_{ij}(-1)$. It follows from the Steinberg relations (i), (ii), and (iii) that both $\rho w_{ij}(u)$ and $\rho h_{ij}(u)$ are monomial matrices in $\mathrm{EL}(n, R)$. In fact the former can be written as PD, with P the permutation matrix describing interchange of the indices i and j, and $D = d(v_1, \ldots, v_n)$ the diagonal matrix with $v_i = -u^{-1}$, $v_j = u$, and $v_k = 1$ for $k \neq i, j$, while $\rho h_{ij}(u) = d_{ij}(u) = d(z_1, \ldots, z_n)$ where the only entries $\neq 1$ are u in the ith spot and u^{-1} in the jth. In this notation, conjugation by $w_{ij}(u)$ and $h_{ij}(u)$ in $\mathrm{ST}(n, R)$ is given by the formulas [13, Corollary 9.4]

$$^{w_{ij}(u)}x_{kl}(r) = x_{\pi(kl)}(v_k r v_l^{-1}), \qquad ^{h_{ij}(u)}x_{kl}(r) = x_{kl}(z_k r z_l^{-1}),$$

where $\pi(kl)$ is the pair of indices obtained from k, l by interchanging i and j. As a consequence, $w_{ji}(u) = w_{ij}(-u^{-1})$ [13, Lemma 9.5].

We now assume that A is a unital algebra over a field k; this guarantees the existence of central units needed in our arguments.

(II) For $n \geqslant 3$, every Steinberg generator $x_{ij}(a)$, $a \in A$, is a commutator by (ii). In case $n = 2$, write $[h_{ij}(u), x_{ij}(a)] = x_{ij}(uau - a)$ for any unit u. Thus $\mathrm{ST}(2, A)$ is perfect if there exists a $u \in k^{\bullet}$ with $u^2 \neq 1$, which is the case unless $k = \mathbf{F}_2$ or $\mathbf{F}_3$.

(III) For technical reasons it is expedient to treat the cases $n = 2$ and $n \geqslant 3$ separately.

For $n = 2$ one begins by assuming that one has a central extension $\psi: G \to \mathrm{ST}(n, A)$ and lifting the generators $x_{ij}(a)$ to cleverly chosen $\sigma x_{ij}(a) \in G$. One then shows that these $\sigma x_{ij}(a)$ satisfy the defining relations (i), (ii), and (iii). Hence the section σ determines a homomorphism $\sigma: \mathrm{ST}(n, A) \to G$ such that $\psi\sigma = id$. The reader should carefully analyze Steinberg's original proof in [19, Section 9] (which is in fact given for all Chevalley groups over fields); we only present the necessary modifications for algebras.

To begin with, one has to require a central unit u with $u^2 - 1 \in A^{\bullet}$, which again excludes the two smallest fields. Next one constructs a function $f: A \times A \to \mathrm{Ker}\, \rho$ with $f(a + a', b) = f(a, b) f(a', b)$, $f(a, b + b') = f(a, b) f(a, b')$

and $f(zaz, zbz) = f(a, b)$ for every $a, a', b, b' \in A$ and $z \in A^{\bullet}$, and one must show that this central function is trivial. Now if $2 \in k^{\bullet}$, then $f(a, b) = f(4a, 4b) = f(a, b)^{16}$, hence $f(15a, b) = 1$ for all $a, b \in A$. Thus if every element of A is divisible by 15, this argument, contributed by A. G. van Asch, shows that $f \equiv 1$. If not, it is enough, in the footsteps of Steinberg, to find a central unit $v \in A^{\bullet}$ such that $v^2 + 1$ is the square of a central unit and $v^4 + v^2 + 1 \in A^{\bullet}$. In characteristic $\neq 2$, according to Steinberg, such a v exists in $k^{\bullet}$ if the field has more than thirteen elements and in characteristic 2, if k has more than four elements. Possible exceptions therefore are algebras over $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$, $\mathbf{F}_5$, $\mathbf{F}_9$.

The proof for $n \geqslant 3$ is modeled after an argument of Kervaire's [11, Proposition 3]; see also Milnor [13, Theorem 5.10] and Steinberg [20, Theorem 14]. However, we have to work a little harder since there the assumption was that $n \geqslant 5$. We begin by listing a number of commutator identities which hold in any group and which we freely make use of:

$$[\alpha, \beta]^{-1} = [\beta, \alpha]; \qquad {}^{\gamma}[\alpha, \beta] = [{}^{\gamma}\alpha, {}^{\gamma}\beta];$$
$$[\alpha\beta, \gamma] = {}^{\alpha}[\beta, \gamma][\alpha, \gamma]; \qquad [\alpha, \beta\gamma] = [\alpha, \beta]^{\beta}[\alpha, \gamma];$$
$$[[\alpha, \beta], \gamma] = [\alpha, \beta][\gamma, \beta][\beta, \gamma\alpha]; \qquad [\alpha, [\beta, \gamma]] = [\alpha\beta, \gamma][\gamma, \alpha][\gamma, \beta],$$

and Philip Hall's identity $[{}^{\gamma}\alpha, [\beta, \gamma]][{}^{\beta}\gamma, [\alpha, \beta]][{}^{\alpha}\beta, [\gamma, \alpha]] = 1$.

Again we assume that $\psi\colon G \to \mathrm{ST}(n, A)$ is a central extension. If $\hat{x}$, $\hat{x}'$ are liftings of elements x, x' in $\mathrm{ST}(n, A)$ to G, the commutator $[\hat{x}, \hat{x}']$ does not depend on the particular choice of these liftings, nor does conjugation in G by such an $\hat{x}$; moreover, $[\hat{x}, \hat{x}']$ is central in G if x and x' commute (Steinberg's trick).

Let $a, b \in A$, then if i, j, k are distinct indices, $f(a, b) = [\hat{x}_{ij}(a), \hat{x}_{ik}(b)]$ defines a central function satisfying $f(a, b + b') = f(a, b) f(a, b')$. Moreover, if $u \in k^{\bullet}$, we have

$${}^{\hat{h}_{jk}(u)\hat{h}_{ik}(u)}f(a, b) = {}^{\hat{h}_{jk}(u)}[\hat{x}_{ij}(ua), \hat{x}_{ik}(u^2 b)] = [\hat{x}_{ij}(a), \hat{x}_{ik}(u^3 b)] = f(a, u^3 b),$$

so $f(a, (1 - u^3)b) = 1$. Hence if k contains a unit u with $u^3 \neq 1$, the function $f \equiv 1$; this is the case when $k \neq \mathbf{F}_2$, $\mathbf{F}_4$; then also $[\hat{x}_{ji}(a), \hat{x}_{ki}(b)] = 1$.

If $n \geqslant 4$, we need not avoid these two fields since working out Hall's identity with $\alpha = \hat{x}_{ij}(a)$, $\beta = \hat{x}_{il}(1)$, $\gamma = \hat{x}_{lk}(b)$ yields $f(a, b) = 1$.

Next $[\hat{x}_{ij}(a), \hat{x}_{ij}(b)] = [\hat{x}_{ij}(a), [\hat{x}_{ik}(1), \hat{x}_{kj}(b)]] = 1$ by taking $k \neq i, j$.

In case $n \geqslant 4$ and i, j, k, l are four distinct indices, write $f(a, b) = f_{ij,kl}(a, b) = [\hat{x}_{ij}(a), \hat{x}_{kl}(b)]$; then $f(a, b + b') = f(a, b) f(a, b')$. Hall's identity with $\alpha = \hat{x}_{ij}(a)$, $\beta = \hat{x}_{kj}(b)$, and $\gamma = \hat{x}_{jl}(c)$ yields $[\hat{x}_{il}(-ac)\hat{x}_{ij}(a), \hat{x}_{kl}(bc)] \cdot [\hat{x}_{kj}(b), \hat{x}_{il}(-ac)] = 1$ or ${}^{\hat{x}_{il}(-ac)}f_{ij,kl}(a, bc) \cdot f_{kj,il}(b, -ac) = 1$. But $f_{ij,kl}(a, bc)$ and $f_{kj,il}(b, -ac)$ are central and ${}^{\hat{w}_{ik}(1)}f_{kj,il}(b, -ac) = [\hat{x}_{ij}(b), \hat{x}_{kl}(ac)] = f_{ij,kl}(b, ac)$, so we find $f(a, bc) = f(b, -ac)$ for all $a, b, c \in A$. Taking $a = 1$, we see that it is enough to prove that $f(1, d) = 1$ for all $d \in A$. Now $f(1, d) = f(1, -d)$ so $f(1, 2d) = 1$. In characteristic $\neq 2$, one concludes that $f \equiv 1$. Otherwise take $u \in k^{\bullet}$ and observe that ${}^{\hat{h}_{kj}(u)\hat{h}_{ki}(u)}f(1, d) = {}^{\hat{h}_{kj}(u)}[\hat{x}_{ij}(u^{-1}), \hat{x}_{kl}(ud)] = [\hat{x}_{ij}(1), \hat{x}_{kl}(u^2d)] = f(1, u^2d)$, hence $f(1, (1-u^2)d) = 1$. If k contains a unit u with $u^2 \neq 1$, we are through. Thus $\mathbf{F}_2$ remains the only possible exception. In case $n \geqslant 5$, we may take a fifth index m and remark that $f(a, b) = [\hat{x}_{ij}(a), [\hat{x}_{km}(1), \hat{x}_{ml}(b)]] = 1$ by putting $\alpha = \hat{x}_{ij}(a)$, $\beta = \hat{x}_{km}(1)$, and $\gamma = \hat{x}_{ml}(b)$ in Hall's identity, so then $\mathbf{F}_2$ need not be excluded.

We have shown that those elements $\hat{x}_{ij}(a)$ which we want to commute, actually do so; we now choose particular liftings $\sigma x_{ij}(a) = y_{ij}(a) = [\hat{x}_{ik}(1), \hat{x}_{kj}(a)]$ and wish to show that these elements satisfy the Steinberg identities (i) and (ii), which establishes a splitting $\sigma\colon \mathrm{ST}(n, A) \to G$ as before. That, in case $n \geqslant 4$, the $y_{ij}(a)$'s do not depend on the choice of the third index k, is seen from the conjugation ${}^{\hat{w}_{lk}(1)}[\hat{x}_{ik}(1), \hat{x}_{kj}(a)] = [\hat{x}_{il}(1), \hat{x}_{lj}(a)]$, since ${}^{\hat{w}_{lk}(1)}y_{ij}(a) = y_{ij}(a)$ by the previous discussion, except perhaps if $n = 4$ and $k = \mathbf{F}_2$.

To prove relation (i), write down the identity $1 = [\hat{x}_{kj}(a), [\hat{x}_{kj}(b), \hat{x}_{ik}(1)]] = [\hat{x}_{kj}(a)\hat{x}_{kj}(b), \hat{x}_{ik}(1)] \cdot [\hat{x}_{ik}(1), \hat{x}_{kj}(a)] \cdot [\hat{x}_{ik}(1), \hat{x}_{kj}(b)]$; since $[\hat{x}_{kj}(a)\hat{x}_{kj}(b), \hat{x}_{ik}(1)] = [\hat{x}_{kj}(a+b), \hat{x}_{ik}(1)]$, this yields $y_{ij}(a)\, y_{ij}(b) = y_{ij}(a+b)$. To prove that $[y_{ij}(a), y_{jk}(b)] = y_{ik}(ab)$, define $f(a, b) = [\hat{x}_{ij}(a), \hat{x}_{jk}(b)] \cdot [\hat{x}_{ij}(1), \hat{x}_{jk}(ab)]^{-1}$. Since both factors commute, it is easily seen that $f(a, b+b') = f(a, b) f(a, b')$. Furthermore, if $u \in k^{\bullet}$, we have

$$ {}^{\hat{h}_{ik}(u)\hat{h}_{jk}(u)}f(a, b) = {}^{\hat{h}_{ik}(u)}f(u^{-1}a, u^2b) = f(a, u^3b) $$

so $f(a, (1-u^3)b) = 1$. Therefore $f \equiv 1$ if k contains a unit u with $u^3 \neq 1$; we only need exclude the fields $\mathbf{F}_2$ and $\mathbf{F}_4$. If $n \geqslant 4$, we can work out Hall's identity with $\alpha = \hat{x}_{ij}(a)$, $\beta = \hat{x}_{jl}(1)$, $\gamma = \hat{x}_{lk}(b)$ to obtain $f(a, b) = 1$, so then the two small fields cause no trouble.

Hence possible exceptions are $\mathrm{ST}(3, A)$ for $\mathbf{F}_2$ and $\mathbf{F}_4$ and $\mathrm{ST}(4, A)$ for $\mathbf{F}_2$. Putting everything together we have proved:

THEOREM 1. *Let A be a unital algebra over a field k. Then* $\mathrm{ST}(n, A)$ *is perfect for every A unless $n = 2$ and $k = \mathbf{F}_2$ or $\mathbf{F}_3$. Moreover,* $\mathrm{ST}(n, A)$ *is centrally closed for every A unless*

$$ \begin{array}{lll} n = 2 & \text{and} & k = \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5, \mathbf{F}_9; \\ n = 3 & \text{and} & k = \mathbf{F}_2, \mathbf{F}_4; \\ n = 4 & \text{and} & k = \mathbf{F}_2. \end{array} $$

This was proved by Steinberg when $A = k$ and according to his results $\mathrm{ST}(n, \mathbf{F}_q)$ is definitely not centrally closed in these exceptional cases [19, p. 116; 20, p. 95], except for $\mathrm{ST}(2, \mathbf{F}_5)$ which is. To justify the inclusion of this field in our list, I exhibit an algebra A over $\mathbf{F}_5$ for which $\mathrm{ST}(n, A)$ is not centrally closed.

However, we postpone this until Section 3 since certain generalities about relative groups from the next section can then be used in the proof without comment.

2. Functors on Algebras

Having dealt with universal central extensions, we now turn to universal coverings. To discuss these, we must first recall how to extend the domain of functors which are defined on unital rings, which depends on the notion of relative groups. Let $F\colon \mathrm{Rg}_1 \to \mathrm{Gr}$ be a functor from unital rings to groups. Given a two-sided ideal $\mathfrak{a}$ in a unital ring R, we write $F(R, \mathfrak{a})$ for the normal subgroup $\mathrm{Ker}(F(R) \to F(R/\mathfrak{a}))$. If the ring homomorphism $R \to R/\mathfrak{a}$ admits a section, the group $F(R)$ is a semidirect product of $F(R/\mathfrak{a})$ with $F(R, \mathfrak{a})$, the former acting on the latter by conjugation. Here are a few examples of relative groups which we need in Section 6.

1. $F = \mathrm{GL}_n$. The group $\mathrm{GL}(n, R, \mathfrak{a})$ consists of all invertible n-square matrices $I + M$ with I the identity matrix and M a matrix with entries in $\mathfrak{a}$. Thus the group $\mathrm{GL}(n, R, \mathfrak{a})$ depends solely on $\mathfrak{a}$ and is sometimes called $\mathrm{GL}(n, \mathfrak{a})$. In this way, GL_n is left exact in that a short exact sequence $\mathfrak{a} \rightarrowtail R \twoheadrightarrow R/\mathfrak{a}$ gives rise to an exact sequence of groups

$$1 \to \mathrm{GL}(n, \mathfrak{a}) \to \mathrm{GL}(n, R) \to \mathrm{GL}(n, R/\mathfrak{a}).$$

2. $F = \mathrm{ST}_n$. For $n \geqslant 3$, $\mathrm{ST}(n, R, \mathfrak{a})$ is the normal subgroup N of $\mathrm{ST}(n, R)$ generated by the conjugates of all $x_{ij}(a)$, $a \in \mathfrak{a}$. Clearly $N \lhd \mathrm{ST}(n, R, \mathfrak{a})$ and to show that the two groups are equal it suffices to construct an inverse to the homomorphism $\mathrm{ST}(n, R)/N \to \mathrm{ST}(n, R/\mathfrak{a})$. This is done by sending $x_{ij}(\bar{r})$ to $x_{ij}(r) \bmod N$ if r is a representative of the residue class $\bar{r} \in R/\mathfrak{a}$, which determines the image of such a generator unambiguously. Relations (i) and (ii) are easily seen to be preserved, so the map is a homomorphism of groups as desired. In case $n = 2$, we must require in addition that every unit in $R/\mathfrak{a}$ can be lifted to a unit in R, which ensures that relation (iii) is preserved. In general, the group $\mathrm{ST}(2, R)/N$ is generated by all elements $\bar{x}_{ij}(\bar{r})$, $\bar{r} \in R/\mathfrak{a}$, subject to relation (i) and and the relation ${}^{w_{ij}(\bar{u})}\bar{x}_{ji}(\bar{r}) = \bar{x}_{ij}(-\bar{u}\bar{r}\bar{u})$ whenever $\bar{u} \in \mathrm{Im}(R^{\boldsymbol{\cdot}} \to (R/\mathfrak{a})^{\boldsymbol{\cdot}})$. Here $w_{ij}(\bar{u})$ of course stands for $\bar{x}_{ij}(\bar{u})\, \bar{x}_{ji}(-\bar{u}^{-1})\, \bar{x}_{ij}(\bar{u})$.

The units of R certainly map onto those of $R/\mathfrak{a}$ if $R \to R/\mathfrak{a}$ admits a section. In this case we can do a little better. Relation (i) allows us to write every generator of $\mathrm{ST}(n, R)$ as a product $x_{ij}(s)\, x_{ij}(a)$ with $a \in \mathfrak{a}$ and s in a subring of S of R which is isomorphic to $R/\mathfrak{a}$. Now, putting $x_{ij}^{(\mu)} = x_{i_\mu j_\mu}$, if $\prod_{\mu=1}^m x_{ij}^{(\mu)}(s_\mu)\, x_{ij}^{(\mu)}(a_\mu) = y \in \mathrm{ST}(n, R, \mathfrak{a})$, then $\prod_{\mu=1}^m x_{ij}^{(\mu)}(s_\mu) = 1$ and we may write

$$y = \prod_{\mu=1}^{m} {}^{z_\mu}x_{ij}^{(\mu)}(a_\mu) \qquad \text{with} \qquad z_\mu = \prod_{\lambda=1}^{\mu} x_{ij}^{(\lambda)}(s_\lambda).$$

Thus $\mathrm{ST}(n, R, \mathfrak{a})$ is generated by the elements $x_{ij}(a)$ under conjugation by $\mathrm{ST}(n, R/\mathfrak{a})$.

The functor ST_n is right exact in that $\mathrm{ST}(n, R) \to \mathrm{ST}(n, R/\mathfrak{a})$ is surjective.

3. $F = \mathrm{EL}_n$. In general, $\mathrm{EL}(n, R, \mathfrak{a})$ is larger than the normal subgroup of $\mathrm{EL}(n, R)$ generated by the $e_{ij}(a)$, $a \in \mathfrak{a}$, even for $n \geqslant 3$. This is the reason why in Section 6 we have to work with the Steinberg group rather than the elementary group. But if $R \to R/\mathfrak{a}$ is split, we can reason as above and show that $\mathrm{EL}(n, R, \mathfrak{a})$ is generated by the $e_{ij}(a)$ under conjugation by $\mathrm{EL}(n, R/\mathfrak{a})$ and therefore under conjugation by $\mathrm{EL}(n, R)$, so then our definition of $\mathrm{EL}(n, R, \mathfrak{a})$ agrees with the usual one [1, V. 1]. The functor EL_n is also right exact.

4. $F = K_2$. From the appropriate sequence defining $K_2(n, R, \mathfrak{a})$ we see that this group may also be described as $\mathrm{Ker}(\mathrm{ST}(n, R, \mathfrak{a}) \to \mathrm{EL}(n, R, \mathfrak{a})) = K_2(n, R) \cap \mathrm{ST}(n, R, \mathfrak{a})$. If $R \to R/\mathfrak{a}$ splits, there is a short exact sequence

$$1 \to K_2(n, R, \mathfrak{a}) \to \mathrm{ST}(n, R, \mathfrak{a}) \to \mathrm{EL}(n, R, \mathfrak{a}) \to 1.$$

For the functors ST_n, EL_n and $K_{2,n}$ the kernel $F(R, \mathfrak{a})$ in general depends on R as well as $\mathfrak{a}$, even when there exists a splitting, i.e., there is no excision [23].

These matters are discussed in a more general framework by Stein [16].

We work with algebras over a fixed commutative ring k with an identity element 1. In our terminology, a k-algebra is a ring A which is a unital k-bimodule with the additional condition that $(ab)c = a(bc)$ whenever one of these three elements is in k and the other two in A. Moreover, the operators from k should commute with all algebra elements. Such an algebra is called unital if A has an identity element e such that the bimodule structure is determined by a unital ring homomorphism η of k into the center of A given by $\eta(u) = ue$, $u \in k$. Furnished with the appropriate morphisms, these algebras form the categories Alg_k resp. Alg_{k1}. An advantage of working in Alg_k is that every morphism has a kernel; this category is punctured, with the null algebra as initial and terminal object. In Alg_{k1} the algebra k is an initial object. A unital algebra A is called augmented if one is given a fixed morphism $\tau: A \to k$ in Alg_{k1}. In other words, τ is a homomorphism of unital rings such that $\tau\eta$ is the identity on k. Then η embeds k into the center of A, viewed in Alg_{k1}, while the augmentation ideal $\mathrm{Ker}\, \tau$ is a k-algebra. The category of augmented k-algebras is a nonfull subcategory of Alg_{k1}. This category Augm_k is punctured, since its object k is both initial and terminal, playing the role of $*$ in the category $\mathrm{Top}*$ discussed in the Introduction.

If A is a k-algebra, we consider pairs (a, u), $a \in A$, $u \in k$, defining addition componentwise and multiplication by the rule $(a, u)(b, v) = (ab + av + ub, uv)$. These pairs then form a unital k-algebra A_k with the structural morphism given by $\eta(u) = (0, u)$, the identity of A_k being the element $(0, 1)$. The algebra A_k admits an augmentation described by $\tau(a, u) = u$ and its augmentation ideal is the k-algebra consisting of all pairs $(a, 0)$, which is of course isomorphic to A. It is now easily seen that the functor $T\colon \mathrm{Alg}_k \to \mathrm{Alg}_{k1}$ defined by $TA = A_k$ is a left adjoint of the (nonfull) embedding $S\colon \mathrm{Alg}_{k1} \to \mathrm{Alg}_k$, the adjunction $I \to ST$ being provided by the map $A \to A_k$ $(a \mapsto (a, 0))$ and $TS \to I$ by $A_k \to A$ $((a, u) \mapsto a + ue)$. Furthermore, if we take T to land in augmented k-algebras, we obtain an equivalence $T\colon \mathrm{Alg}_k \to \mathrm{Augm}_k$ of punctured categories, its pseudo-inverse being given by attaching to every augmented k-algebra its augmentation ideal. Suppose now that $A \in \mathrm{Alg}_{k1}$ with identity element e. Then $TSA = A_k$ is isomorphic in Alg_{k1} with $A \times k$ by the mapping $(a, u) \mapsto (a + ue) \times u$. This is even an isomorphism of augmented algebras where the augmentation $\tau\colon A \times k \to k$ is defined by projecting onto the second factor. Notice that if $k = \mathbf{Z}$, we are just working with rings, since $\mathrm{Alg}_{\mathbf{Z}} = \mathrm{Rg}$, $\mathrm{Alg}_{\mathbf{Z}1} = \mathrm{Rg}_1$ and embedding A in $TA = A_{\mathbf{Z}}$ (often denoted by A^+) is the familiar process of adjoining an identity element to a ring. That case is described in [22, Section 8], of which the present discussion is a straightforward generalization to algebras, designed to fix terminology and notation.

If A is a k-algebra and F a functor from Alg_{k1} to Gr, we know how to describe $F(A_k, A) = \mathrm{Ker}(F\tau\colon F(A_k) \to F(k))$ as a normal subgroup of $F(A_k)$ upon which $F(k)$ acts by conjugation. Now if A is a unital algebra, $F(A_k, A)$ may be identified with $F(A)$ provided $F(A \times k) = F(A) \times F(k)$. If this is true for all $A \in \mathrm{Alg}_{k1}$, defining $F(A) = F(A_k, A)$ yields an unambiguous extension of the functor F to the category Alg_k. The functors GL_n and EL_n, given as matrix groups, clearly preserve finite products. The Steinberg functor ST_n, hence $K_{2,n}$, does this for $n \geqslant 3$, but not in full generality for $n = 2$. This question occupies us in Section 7.

An important observation is that the extended functor GL_n remains left exact on Alg_k. Indeed, if $0 \to A \to B \xrightarrow{f} C \to 0$ is an exact sequence of algebras, we also have an exact sequence $0 \to A \to B_k \xrightarrow{Tf} C_k \to 0$ in Alg_{k1}, whence $1 \to \mathrm{GL}(n, A) \to \mathrm{GL}(n, B_k) \to \mathrm{GL}(n, C_k)$ is exact. Take the augmentations of B_k and C_k and apply GL_n. One sees that the sequence of kernels $1 \to \mathrm{GL}(n, A) \to \mathrm{GL}(n, B) \to \mathrm{GL}(n, C)$ is exact, again using left exactness of GL_n with respect to ideals in unital rings.

3. A Digression

We now come to the promised example of an algebra A over $\mathbf{F}_5$ for which $\mathrm{ST}(2, A)$ is not centrally closed.

We first introduce some notation, which is in force throughout. If $pq = qp = 0$ for two elements in a unital ring R, we write $\langle p, q\rangle_{ij} = [x_{ji}(-q), x_{ij}(p)]$; if u and v are commuting units we write $\{u, v\}_{ij} = h_{ij}(uv)\, h_{ij}(u)^{-1} h_{ij}(v)^{-1}$. If $n \geqslant 3$, neither of these symbols depends on the pair of indices i, j; for $n \geqslant 2$, both these symbols are mapped to the identity under ρ, hence live in $K_2(n, R)$; the element

$\{u, v\}_{ij}$ is always central in $\mathrm{ST}(n, R)$ and so is the element $\langle p, q\rangle_{ij}$ for $n \geqslant 3$ [7, Section 9].

LEMMA 2. *With p and q as above, suppose $1 + p$ and $1 + q$ are units. Then* $\langle p, q\rangle_{ji} = \{1 + p, 1 + q\}_{ij}$.

The lemma follows by conjugating the central identity $[x_{ji}(p), x_{ij}(q)] = \{1 + p, 1 + q\}_{ij}$ with $x_{ij}(-q)$. For commutative rings this useful identity occurs as [18, Corollary 2.9]. A direct although shorter derivation still requires a tedious calculation, which I omit. One only needs to work within the commutative subring of R generated by $1, p$, and q.

Now let k be a commutative ring and $\epsilon^2 = 0$; the dual numbers $k[\epsilon]$ form an augmented k-algebra. The Steinberg group $\mathrm{ST}(n, k[\epsilon])$ is a semidirect product of $\mathrm{ST}(n, k)$ and the normal subgroup $\mathrm{ST}(n, k[\epsilon], (\epsilon))$. The former acts upon the latter by conjugation; see Section 2.

PROPOSITION 3. *If $k \neq \mathbf{F}_9$ is a finite field of odd characteristic p, then $\mathrm{ST}(2, k[\epsilon]) = \mathrm{EL}(2, k[\epsilon])$ and $\mathrm{ST}(2, k[\epsilon], [\epsilon)) = \mathrm{EL}(2, k[\epsilon], (\epsilon))$ is an Abelian normal subgroup isomorphic to $k^+ \times k^+ \times k^+$, where k^+ is the additive group of the field.*

Proof. By a result of Stein [18, Theorem 2.13], $K_2(2, k[\epsilon])$ is generated by the elements $\{u, v\}_{ij}$, all of which are central in $\mathrm{ST}(2, k[\epsilon])$; see also Corollary 6 and Theorem 7 to follow. We show that $\{a + b\epsilon, c + d\epsilon\}_{ij} = 1$ for all $a, c \in k$, $b, d \in k$, using the relations satisfied by Steinberg cocycles $\{\ ,\ \}_{ij}$ which are conveniently summarized in [18, Proposition 1.1, (S1)–(S8)]. In fact, our proof is close to certain arguments in that paper.

First recall that Steinberg had already shown that $\mathrm{ST}(2, k) = \mathrm{EL}(2, k)$ for a finite field, hence $\{a, c\}_{ij} = 1$ for $a, c \in k^{\boldsymbol{\cdot}}$ [19, Theorem 3.3]. Next, the commutator identity $[\alpha\beta, \gamma] = {}^{\alpha}[\beta, \gamma][\alpha, \gamma]$ and the centrality of $\langle b\epsilon, d\epsilon\rangle_{ji}$ guaranteed by the preceding lemma, shows that this expression is additive in b. Similarly it is additive in d (though of course we write the operation in ST_2 multiplicatively). Hence $\langle b\epsilon, d\epsilon\rangle_{ji}^{p} = 1$.

Suppose $k = \mathbf{F}_q$, $q = p^s$, and choose a generator y for the multiplicative group $k^{\boldsymbol{\cdot}}$ which has order $q - 1$. Since $q \neq 9$, we know that every element of k is a sum of fourth powers [18, Lemma 3.6]. By additivity, $\langle b\epsilon, d\epsilon\rangle_{ji}$ is 1 if we can prove this for $b = y^{4m}$, $d = y^{4n}$ with arbitrary integers m and n. Let $r = m - n$ and write $u = y^r \in k^{\boldsymbol{\cdot}}$. Then, using (iii), we see that

$$\begin{aligned}\langle b\epsilon, d\epsilon\rangle_{ji} &= {}^{w_{ji}(u)}\langle b\epsilon, d\epsilon\rangle_{ji} = {}^{w_{ji}(u)}[x_{ij}(-d\epsilon), x_{ji}(b\epsilon)] \\ &= [x_{ji}(u^2 d\epsilon), x_{ij}(-u^{-2}b\epsilon)] = \langle -u^{-2}b\epsilon, -u^2 d\epsilon\rangle_{ij} = \langle f\epsilon, f\epsilon\rangle_{ij}\end{aligned}$$

with $f = y^t$ where $t = 4m - 2r = 4n + 2r$. Now $\langle f\epsilon, f\epsilon\rangle_{ij}^2 = \langle f\epsilon, 2f\epsilon\rangle_{ij} =$

$\{1+f\epsilon, 1+2f\epsilon\}_{ji} = \{1+f\epsilon, (1+f\epsilon)^2\}_{ji} = 1$ by (S8). We conclude that $\{1+b\epsilon, 1+d\epsilon\}_{ij} = \langle b\epsilon, d\epsilon\rangle_{ji}$, having order p and order 2, must be 1.

In the group of units $k[\epsilon]^\bullet$ any unit $a \in k^\bullet$ has order dividing $q-1$, while the order of $1+\frac{1}{2}d\epsilon$ is p; hence the subgroup of $k[\epsilon]^\bullet$ generated by these two units is cyclic. Dropping the subscript ij of the cocycle, we then derive successively:

$$\{a, 1+c^{-1}d\epsilon\} = \{a, (1+\tfrac{1}{2}c^{-1}d\epsilon)^2\} = 1 \qquad \text{using (S7) and (S8);}$$
$$\{a+b\epsilon, 1+c^{-1}d\epsilon\} = \{a, 1+c^{-1}d\epsilon\}\cdot\{1+a^{-1}b\epsilon, 1+c^{-1}d\epsilon\} = 1 \qquad \text{using (S6)}$$
$$\{a+b\epsilon, c\} = \{1+a^{-1}b\epsilon, c\}\cdot\{a, c\} = 1 \qquad \text{using (S6)}$$

and finally

$$\{a+b\epsilon, c+d\epsilon\} = \{a+b\epsilon, 1+c^{-1}d\epsilon\}\cdot\{a+b\epsilon, c\} = 1 \qquad \text{using (S6).}$$

This proves the first half of the proposition.

For any commutative ring k, the group $\mathrm{EL}(2, k[\epsilon], (\epsilon))$ consists of all matrices

$$M = \begin{pmatrix} 1-r\epsilon & -s\epsilon \\ t\epsilon & 1+r\epsilon \end{pmatrix} \qquad \text{with} \qquad r, s, t \in k.$$

Such matrices commute, and $M = e_{12}(-s\epsilon)d_{12}(1-r\epsilon)e_{21}(t\epsilon)$, where $d_{12}(1-r\epsilon) = \mathrm{diag}(1-r\epsilon, 1+r\epsilon)$. This is a unique decomposition within the group. The collections of matrices of type $e_{12}(-s\epsilon)$, $d_{12}(1-r\epsilon)$, and $e_{21}(t\epsilon)$, respectively, form subgroups isomorphic to k^+. Hence $\mathrm{EL}(2, k[\epsilon], (\epsilon)) \simeq k^+ \times k^+ \times k^+$. Since for our finite field k the normal subgroup $\mathrm{ST}(2, k[\epsilon], (\epsilon))$ equals this matrix group by the first part, the full proposition has been proved.

The action of $\mathrm{EL}(2, k)$ on $\mathrm{EL}(2, k[\epsilon], [\epsilon))$ is conveniently described by writing the latter additively. In particular, if k is a prime field of positive characteristic, $\mathrm{EL}(2, k[\epsilon], (\epsilon))$ is a three-dimensional vector space over k with basis $e_{12}(-\epsilon)$, $d_{12}(1-\epsilon)$, $e_{21}(\epsilon)$. The action by conjugation is then seen to be given by

$$e_{12}(a) \mapsto \begin{pmatrix} 1 & -2a & a^2 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}, \qquad e_{21}(b) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ -b & 1 & 0 \\ b^2 & -2b & 1 \end{pmatrix},$$

$a, b \in k$. That this really defines a representation σ of $\mathrm{EL}(2, k)$ is independently checked by verifying that σ preserves the Steinberg relations (i) and (ii). In fact, for any finite field, this is the irreducible three-dimensional modular representation determined by the homogeneous polynomial $x_0^2 - x_0 y_0 + y_0^2$ in the treatment of [2, Section 30], the adjoint representation in the terminology of algebraic groups.

We are now ready for the promised

EXAMPLE 4. The group $\mathrm{ST}(2, \mathbf{F}_5[\epsilon])$ admits a nontrivial central extension by a 5-cyclic group.

Proof. Put

$$\tau x_{12}(a) = \begin{pmatrix} 1 & a & -a^2 & 2a^3 \\ 0 & 1 & -2a & a^2 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \tau x_{21}(b) = \begin{pmatrix} 1 & -2b^3 & b^2 & -b \\ 0 & 1 & 0 & 0 \\ 0 & -b & 1 & 0 \\ 0 & b^2 & -2b & 1 \end{pmatrix}$$

for $a, b \in \mathbf{F}_5$. The reader may ascertain that this preserves the relations (i) and (iii), hence yields a four-dimensional modular representation τ of $\mathrm{EL}(2, \mathbf{F}_5)$, which contains σ as a constituent. Reverting to the multiplicative point of view, we obtain a group G which is a semidirect product of $\mathrm{EL}(2, \mathbf{F}_5)$ and $(\mathbf{F}_5{}^+)^4$. The new basis vector, say e, is stable under conjugation by $\mathrm{EL}(2, \mathbf{F}_5)$; hence collapsing $e \mapsto 1$ gives rise to a central extension of groups. This appears as the middle row in the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \langle e \rangle \arrow[r] \arrow[d, equal] & (\mathbf{F}_5{}^+)^4 \arrow[r] \arrow[d] & \mathrm{EL}(2, \mathbf{F}_5[\epsilon], (\epsilon)) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \langle e \rangle \arrow[r] & G \arrow[r] \arrow[d] & \mathrm{EL}(2, \mathbf{F}_5[\epsilon]) \arrow[r] \arrow[d] & 1 \\
 & & \mathrm{EL}(2, \mathbf{F}_5) \arrow[r, equal] \arrow[d] & \mathrm{EL}(2, \mathbf{F}_5) \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}$$

It remains to be proved that this extension does not split, or, in terms of the representation, that σ is not a direct factor of τ. One may show this by an argument involving the Jordan normal form of the matrix $\tau e_{12}(1)$ or by verifying that the cocycle governing the extension τ of σ by the trivial representation is not a coboundary. Reasoning as in Steinberg [19, Section 9] one obtains that G is actually a universal central extension of $\mathrm{ST}(2, \mathbf{F}_5[\epsilon]) = \mathrm{EL}(2, \mathbf{F}_5[\epsilon])$, which means that this last group has a 5-cyclic Schur multiplier.

4. THE UNIVERSAL COVERING OF A FUNCTOR

In this section k is again a fixed commutative ring with 1. If X is a set with base point $*$, we denote by $k\langle X\rangle$ the free algebra over k on X. The elements of $k\langle X\rangle$ are polynomials in noncommuting indeterminates, one for each $x \in X$,

$x \neq *$, with coefficients from k. It is expedient to attach the empty polynomial to $*$. An augmentation τ: $k\langle X\rangle \to k$ is given by sending each such indeterminate to 0. The augmentation ideal, consisting of all polynomials without a constant term, we write as $\langle X\rangle$. This is a k-algebra, and the unital k-algebra $k\langle X\rangle$ is just $\langle X\rangle_k$. If X has more than one element $\neq *$, the center of $k\langle X\rangle$ is k; if k is an integral domain, $k\langle X\rangle^{\bullet} = k^{\bullet}$. When F is a group valued functor on Alg_k, we often write $F(\langle X\rangle)$ for $F(k\langle X\rangle, \langle X\rangle)$ since, $\langle X\rangle$ being without identity, no confusion arises.

Now suppose A is a k-algebra. Considering A as a set with base point 0, we can build $k\langle A\rangle$ and are in the habit of writing EA for the augmentation ideal $\langle A\rangle$. Thus $k\langle A\rangle = (EA)_k$. Each indeterminate X_a, attached to $a \in A$, lives in EA; if we map $X_a \mapsto a$, we get a homomorphism of k-algebras ϵ: $EA \to A$. There is also a homomorphism μ: $EA \to E^2A$ defined by $\mu(X_a) = Y_{X_a}$, and all this makes (E, ϵ, μ) a cotriple in the category Alg_k; cf. [21, 7].

In [21, 15] we showed how to attach to a group-valued functor F: $\mathrm{Rg} \to \mathrm{Gr}$ a universal covering functor with respect to a given cotriple in $\mathrm{Rg} = \mathrm{Alg}_{\mathbf{Z}}$. There the theory with respect to the free cotriple was adorned with the superscript I. Since this is the only case we deal with here, we drop the Roman numeral. However, in the balance of this section, we just work with an arbitrary cotriple (E, ϵ, μ) in Alg_k.

The results of [21, Sections 14 and 15] are readily extended to algebras, so we merely state the relevant facts without proofs, referring to the earlier paper for details as well as motivation from topology: cf. also the Introduction.

Given a functor F: $\mathrm{Alg}_k \to \mathrm{Gr}$ (the reader may like to think of an affine group scheme over k), we begin by calling FE: $\mathrm{Alg}_k \to \mathrm{Gr}$ the "paths with base point" on F and $F\epsilon: FE \to F$ the operation of "taking end points." The "loop space" on F is then the fiber $\mathrm{Ker}\, F\epsilon$, and $\mathrm{Im}\, F\epsilon = F^0$ is the "connected component" of F. We define "coverings" λ: $H \to F$, H: $\mathrm{Alg}_k \to \mathrm{Gr}$, with respect to "unique lifting of paths." In fact these cover F^0. It is shown in [21, Theorem 15.3] that every F has a "universal covering" $\Lambda: \hat{F} \to F$, of course unique up to isomorphism. Here $\mathrm{Im}\, \Lambda$ is the connected component of F and the exact sequence

$$1 \longrightarrow \pi_1 F \longrightarrow \hat{F} \xrightarrow{\Lambda} F \longrightarrow \pi_0 F \longrightarrow 1 \tag{A}$$

defines the "homogeneous space of components" $\pi_0 F$ and the "fundamental group" $\pi_1 F$ of F. The functors in the exact sequence all go from $\mathrm{Alg}_k \to \mathrm{Gr}$, except for $\pi_0 F$, which in general only lands in sets with base point.

The construction of $\hat{F}$ is best understood from the following exact ladder of functors $\mathrm{Alg}_k \to \mathrm{Gr}$:

$$\begin{tikzcd} 1 \arrow[r] & M_2 \arrow[r] \arrow[d, "d"] & FE^2 \arrow[r, "FE\epsilon"] \arrow[d, "F\epsilon E"] & FE \arrow[r] \arrow[d, "F\epsilon"] & 1 \\ 1 \arrow[r] & Y_1 \arrow[r] & FE \arrow[r, "F\epsilon"'] & F & \end{tikzcd} \tag{B}$$

where M_2 and Y_1 are defined as kernels of $FE\epsilon$ and $F\epsilon$, respectively, and d is the restriction of $F\epsilon E$ to M_2. It turns out that $\operatorname{Im} d$ is normal in FE (for each k-algebra) and $\hat{F}$ is defined as $FE/\operatorname{Im} d$. Factoring out $\operatorname{Im} d$ from the bottom row then yields the sequence (A) which defines the functors $\pi_i F$, $i = 0, 1$. As one would expect, $\pi_0 \hat{F} = \pi_1 \hat{F} = 1$, i.e., the universal covering is simply connected. It is also explained in [21, p. 266] that the $\pi_1 F$ so defined is none other than the simplicial fundamental group of F using the standard cotriple resolution of E.

5. Presentations for some Groups

In order to come to grips with the fundamental group, we must study the behavior of our functors on free algebras. We treat this in the more general framework of obtaining presentations for certain groups. Throughout this section, R is a unital ring.

Following [3; 14] the subgroup of $\mathrm{GL}(n, R)$ generated by $\mathrm{EL}(n, R)$, and all diagonal matrices $d(u_1, \ldots, u_n)$, $u_i \in R$, is called $\mathrm{GE}(n, R)$. The elementary matrices clearly satisfy the counterparts (i)′, (ii)′, (iii)′ of the Steinberg relations for the images $e_{ij}(r) = \rho x_{ij}(r)$. The diagonal matrices satisfy

$$\text{(iv)}' \quad d(u_1, \ldots, u_n)\, d(v_1, \ldots, v_n) = d(u_1 v_1, \ldots, u_n v_n)$$

while conjugation of an elementary matrix by a diagonal one is described by

$$^{d(u_1, \ldots, u_n)}e_{ij}(r) = e_{ij}(u_i r u_j^{-1}). \tag{v$'$}$$

Writing as in Section 1, for $u \in R^{\bullet}$, $\rho h_{ij}(u) = d_{ij}(u)$, the matrix which only differs from the identity in the ith spot on the diagonal where the entry is u and the jth where it is u^{-1}, one also has the relation

$$\text{(vi)}' \quad e_{ij}(u-1)\, e_{ij}(1)\, e_{ij}(u^{-1}-1)\, e_{ji}(-u) = d_{ij}(u),$$

which secures these particular diagonals as products of elementaries.

Thus (i)′ through (vi)′ are the "obvious" relations which hold for every ring, and a ring R for which (i)′ $\cdots$ (vi)′ give a presentation of $\mathrm{GE}(n, R)$ is called universal for GE_n. Such rings were investigated by Cohn for $n = 2$ [3] and by Silvester for $n \geqslant 2$ [14; 15].

Write $H(n, R)$ for the subgroup of $\mathrm{ST}(n, R)$ generated by all elements $h_{ij}(u)$, $1 \leqslant i \neq j \leqslant n$, $u \in R^{\bullet}$.

Theorem 5. (R.K. Dennis) *The ring R is universal for GE_n if and only if*

(a) $K_2(n, R) \subset H(n, R)$

and

(b) $d(v, 1, \ldots, 1) \in \mathrm{EL}(n, R)$ *implies v is a unit in the commutator subgroup* $[R^{\bullet}, R^{\bullet}]$.

This is proved in an unpublished paper of Dennis [5], by means of the Reidemeister–Schreier method of obtaining a presentation for a subgroup, and announced in [8, p. 291]. An easier argument was then suggested by Swan, which was kindly communicated to me by Dennis. In order to present this, we first make a few quick observations on groups with operators.

Suppose D, H, S are groups such that

(1) D acts on S and H is an invariant subgroup under this action, which we denote by $*$;

(2) there is a D-homomorphism $\phi: H \to D$, D acting on itself by conjugation;

(3) for $h \in H$ and $s \in S$ we have $\phi(h) * s = {}^h s$.

Under these circumstances, it is easy to verify that $\operatorname{Im}\phi$ is normal in D and $\operatorname{Ker}\phi$ is contained in the center of S. Write $\operatorname{Im}\phi = D_0 \lhd D$.

Now consider the semidirect product $S * D_0$ defined by $s_1 d_1 \cdot s_2 d_2 = s_1(d_1 * s_2)\, d_1 d_2$, $s_i \in S$, $d_i \in D_0$. Write $\bar{S} = S/\operatorname{Ker}\phi$; then sending sd to $\overline{sh}$ if $\phi(h) = d$ (with obvious notation) is easily seen to define a map $\vartheta: S * D_0 \to \bar{S}$. This turns out to be a homomorphism because ϕ preserves the action of D. Furthermore $\operatorname{Ker}\vartheta$ is a normal subgroup N of $S * D_0$ consisting of all elements $h^{-1}\phi(h)$, $h \in H$. Multiplication in N works out as

$$h_1^{-1}\phi(h_1) \cdot h_2^{-1}\phi(h_2) = h_2^{-1}h_1^{-1}\phi(h_1 h_2) \qquad \text{because of (3).}$$

We may extend the action on S from D_0 to D and form the semidirect product $S * D$. The formula ${}^{sd}(h^{-1}\phi(h)) = d * h^{-1} \cdot \phi(d * h)$ shows that N is also normal in this bigger group. Projection of $S * D$ onto its factor D yields a short exact sequence of groups

$$1 \to \bar{S} \to (S * D)/N \to D/D_0 \to 1.$$

We apply this setup to $S = \mathrm{ST}(n, R)$, $H = H(n, R)$, and $D = D(n, R)$, the group of invertible n-square diagonal matrices over R. Define $d(u_1, \ldots, u_n) * x_{ij}(r) = x_{ij}(u_i r u_j^{-1})$. It is easy to check that this operation preserves the Steinberg relations (i), (ii), (iii), hence establishes an action of D on S. One then readily derives the formula $d(u_1, \ldots, u_n) * h_{ij}(v) = h_{ij}(u_i v u_j^{-1})\, h_{ij}(u_i u_j^{-1})^{-1}$ which shows that H is invariant under the action, proving (1). The surjection $\rho: \mathrm{ST}(n, R) \to \mathrm{EL}(n, R)$ given by $x_{ij}(r) \mapsto e_{ij}(r)$ is manifestly a D-homomorphism, D acting by conjugation on $\mathrm{EL}(n, R)$. Call the restriction to the smaller groups $\phi: H \to D$, securing (2). Condition (3) follows from the formula ${}^{h_{ij}(u)}x_{kl}(r) = x_{kl}(u_k r u_l^{-1})$ already encountered in Section 1, where we have introduced for convenience $d(u_1, \ldots, u_n) = d_{ij}(u) = \phi h_{ij}(u)$.

We find that $\operatorname{Ker}\phi = K_2(n, R) \cap H(n, R) = C(n, R)$ is central in $\mathrm{ST}(n, R)$; it is the group denoted by C_n in [13, Corollary 9.3]. Next observe that $S * D$

is a group generated by all elements $x_{ij}(r)$ and $d(u_1, \ldots, u_n)$. The former satisfy (i), (ii), (iii), the latter relation (iv)′, describing multiplication in S and D, respectively. The semidirect product is determined by relation (v) (simply (v)′) with the e_{ij}'s replaced by x_{ij}'s). Thus (i), (ii), (iii), (iv)′, (v) present $S * D$. As for the normal subgroup N, its multiplication mentioned above shows that it is the subgroup of $S * D$ generated by all elements $h_{ij}(u)^{-1}d_{ij}(u)$.

Finally the group $D_0 = \operatorname{Im} \phi$ consists of all products $d(u_1, \ldots, u_n)$ of elements $d_{ij}(v)$. We claim that $u_1 \cdots u_n \in [R^\bullet, R^\bullet]$. This is trivially true when the product has only a single factor. If $d(u_1, \ldots, u_n) = d(v_1, \ldots, v_n)\, d_{ij}(v)$ then $u_1 \cdots u_n = xvyv^{-1}z$, where x, y, and z are units whose product xyz is in $[R^\bullet, R^\bullet]$ by the induction assumption. The identity $xvyv^{-1}z = [x, v] \cdot {}_k(xyz) \cdot [v, z^{-1}]$ shows that $u_1 \cdots u_n$ is a product of commutators. On the other hand every commutator $[u, v]$ can be realized as $d_{12}(uv)\, d_{12}(u)^{-1}d_{12}(v)^{-1}$; therefore, D_0 consists of exactly those diagonal matrices $d(u_1, \ldots, u_n)$ with $u_1 \cdots u_n \in [R^\bullet, R^\bullet]$. Notice that sending $d(u_1, \ldots, u_n)$ to the product $u_1 \cdots u_n$ induces an isomorphism from D/D_0 to $R^\bullet_{ab}$, the abelianized group of units of R.

Reverting to the group $\mathrm{GE}(n, R)$, observe that $\mathrm{EL}(n, R)$ is a normal subgroup and that $d(u_1, \ldots, u_n) \equiv d(v, 1, \ldots, 1) \bmod \mathrm{EL}(n, R)$ with $v = u_1 \cdots u_n$. Write $V(n, R)$ for the group of units v such that $d(v, 1, \ldots, 1) \in \mathrm{EL}(n, R)$. Clearly $V(n, R)$ contains $[R^\bullet, R^\bullet]$ hence is normal in $R^\bullet$ and we may write $R^\bullet_n = R^\bullet/V(n, R)$. Mapping $d(u_1, \ldots, u_n)$ to $\bar{v} = \overline{u_1 \cdots u_n}$ induces an isomorphism of $\mathrm{GE}(n, R)/\mathrm{EL}(n, R)$ with $R^\bullet_n$, where the bar means working modulo $V(n, R)$.

Proof of Theorem 5. Consider the exact ladder of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \bar{S} & \longrightarrow & (S * D)/N & \longrightarrow & D/D_0 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{EL}(n, R) & \longrightarrow & \mathrm{GE}(n, R) & \longrightarrow & R^\bullet_n & \longrightarrow & 1 \end{array}$$

in which the left-hand column is induced by ρ, the middle one by the map $S * D \to \mathrm{GE}(n, R)$ given by $sd \mapsto \rho(s)d$ and the right-hand one by $d(u_1, \ldots, u_n) \mapsto \bar{v}$ where $v = u_1 \cdots u_n$. That all these give rise to homomorphisms onto which make the diagram commute, is more or less immediate after what went before. We then have an exact sequence of vertical kernels

$$1 \to X \to Y \to Z \to 1.$$

$(X) = (1)$ is equivalent with condition (a):

Indeed, $\operatorname{Ker} \rho \subset \operatorname{Ker} \phi \subset H(n, R)$ means $K_2(n, R) = \operatorname{Ker} \phi = C(n, R)$.

$Z = (1)$ is equivalent with condition (b):

$Z = (1)$ means $R^\bullet_{ab} = R^\bullet_n$ or $V(n, R) = [R^\bullet, R^\bullet]$.

$Y = (1)$ means R is universal for GE_n:

The calculation

$$\begin{aligned} h_{ij}(u) &= w_{ij}(u)\, w_{ij}(-1) = x_{ij}(u)\, x_{ji}(-u^{-1})\, x_{ij}(u)\, w_{ij}(-1) \\ &= x_{ij}(u-1)\, x_{ji}(1)\, x_{ij}(-1) \cdot {}^{w_{ij}(1)}(x_{ji}(-u^{-1})\, x_{ij}(u)) \\ &= x_{ij}(u-1)\, x_{ji}(1)\, x_{ij}(u^{-1}-1)\, x_{ji}(-u) \end{aligned}$$

shows that relation (vi) ((vi)′ with e_{ij}'s replaced by x_{ij}'s) is nothing but $h_{ij}(u)^{-1}\, d_{ij}(u) = 1$. These elements $h_{ij}(u)^{-1} d_{ij}(u)$ generate N and are carried to $d_{ij}(u)^{-1}\, d_{ij}(u) = 1$ under the map $S * D \to \mathrm{GE}(n, R)$. If N is the whole kernel of this homomorphism, this means the group $\mathrm{GE}(n, R)$ is presented by (i)′ through (vi)′, i.e., R is universal for GE_n.

This finishes the proof of Dennis' theorem.

I have been informed since that more general results have been obtained by Geller in her 1975 Cornell thesis, see the announcement [10]. Dennis' theorem is a special case of an exact sequence in that note.

Discussion. Condition (a) is equivalent with $K_2(n, R) \subset W(n, R)$ where $W(n, R)$ is the Weyl group generated by all elements $w_{ij}(r)$ in $\mathrm{ST}(n, R)$. Indeed, one always has [13, Theorem 9.11] $C(n, R) = K_2(n, R) \cap H(n, R) = K_2(n, R) \cap W(n, R)$; the assumptions that R is commutative and $n \geqslant 3$ are not needed in this part of that theorem. If $R^{\bullet}$ is Abelian, $C(n, R)$ is the subgroup of $\mathrm{ST}(n, R)$ generated by all Steinberg cocycles $\{u, v\}_{ij}$, $u, v \in R^{\bullet}$, ibid. Again, this also holds for $n = 2$. If $n \geqslant 3$, these symbols $\{u, v\}_{ij}$ are bimultiplicative and independent of i, j, and are known as Steinberg symbols [13, Lemma 9.7]. Condition (b) is trivially seen to be satisfied for commutative rings by taking determinants. Therefore we have

COROLLARY 6 (R. K. Dennis). *A commutative ring is universal for* GE_n *if and only if* $K_2(n, R)$ *is generated by Steinberg cocycles.*

The following result was proved in [15, Theorem 14]. See also [8, p. 291; 18, Theorem 2.13] the latter of which contains a useful survey. If R is a ring, we write J for its radical and $\mathrm{M}_2(R)$ for the full ring of 2 by 2 matrices over R.

THEOREM 7. *A semilocal ring R is universal for* GE_n, $n \geqslant 3$, *if R/J does not contain* $\mathbf{F}_2 \times \mathbf{F}_2$ *or* $\mathrm{M}_2(\mathbf{F}_2)$ *as a factor. If R is local or if R/J contains neither* $\mathbf{F}_2$ *nor* $\mathrm{M}_2(\mathbf{F}_2)$ *as a direct factor, R is universal for* GE_2.

For semilocal algebras this gives us, together with Theorem 5, the centrality of the Steinberg extension, whose universality was treated in Theorem 1. Hence

THEOREM 8. *For every semilocal unital algebra A over a field k the group*

$\mathrm{ST}(n, A)$ *is the universal central extension and* $K_2(n, A) = H_2(\mathrm{EL}(n, A), \mathbf{Z})$, *the Schur multiplier of* $\mathrm{EL}(n, A)$, *unless*

$$\begin{array}{lll} n = 2 & \text{and} & k = \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5, \mathbf{F}_9; \\ n = 3 & \text{and} & k = \mathbf{F}_2, \mathbf{F}_4; \\ n = 4 & \text{and} & k = \mathbf{F}_2. \end{array}$$

Our aim is to use Silvester's theorem that free algebras $k\langle X\rangle$ over a field k are universal for GE_n to establish that $K_2(n, k\langle X\rangle)$ is generated by the Steinberg cocycles. For $n = 2$, however, we prefer to present a direct demonstration, since we can prove a more general result using an idea of A. Bak.

A degree function on a ring is defined in [4, p. 32] as a mapping from R to the real numbers satisfying certain properties. We stipulate the same properties, but relax the range to be any Abelian totally ordered semigroup G, with neutral element 0 and written additively. This means that if $g_1 < g_2$ in G, then $g_1 + g < g_2 + g$ for every $g \in G$, We adjoin a symbol $-\infty$ with the usual conventions. A mapping $d\colon R \to \{G \cup -\infty\}$ is called a degree function if

$$\begin{aligned} &d(a) \geqslant 0 \qquad \text{for } a \neq 0, \quad d(0) = -\infty; \\ &d(ab) = d(a) + d(b); \\ &d(a - b) \leqslant \max\{d(a), d(b)\} \qquad \text{for all } a \text{ and } b \in R. \end{aligned}$$

Then the ring R can have no zero divisors and $R^{\bullet} \subset R_0$, the subring consisting of all elements with degree $\leqslant 0$. It is readily ascertained that $d(-a) = d(a)$ and if $d(a) \neq d(b)$, then $d(a \pm b) = \max\{d(a), d(b)\}$.

We write $K_2'(2, R_0)$ for the image of $K_2(2, R_0)$ in $K_2(2, R)$ under the embedding $R_0 \subset R$. Frequently this image may be identified with $K_2(2, R_0)$, for instance if R is an augmented R_0-algebra since then $K_2(2, R_0) \to K_2(2, R)$ admits a splitting. Similarly for the images $\mathrm{ST}'(2, R_0)$ and $\mathrm{EL}'(2, R_0)$. The following theorem and its proof are generalizations of a result I learned from A. Bak; see Theorem 10.

THEOREM 9. *If R is a ring with a degree function and* $V(2, R_0) = [R_0^{\bullet}, R_0^{\bullet}]$, *then the group* $K_2(2, R)$ *is normally generated in* $\mathrm{ST}(2, R)$ by $K_2'(2, R_0)$.

Proof. Write N for the normal subgroup of $\mathrm{ST}(2, R)$ generated by $K_2'(2, R_0)$. In order to show that N is the whole of $K_2(2, R)$ we write an element x of $K_2(2, R)$ as a product $x = myn$ with $m \in \mathrm{ST}'(2, R_0)$, $y \in \mathrm{ST}(2, R)$, and $n \in N$. Clearly, this description is very redundant and far from unique. We shall rewrite x as $x = m'y'n'$ in which the element $y' \in \mathrm{ST}(2, R)$ has a shorter expression in the Steinberg generators than y. Continuing by induction, we eventually find $x = m'' \cdot 1 \cdot n''$. Both x and n'' map to the identity matrix under the surjective homomorphism $\rho\colon \mathrm{ST}(2, R) \to \mathrm{EL}(2, R)$ which entails that $\rho(m'') = 1$. Hence $m'' \in K_2'(2, R_0)$ and x is in N.

Each of the factors of x is a product of Steinberg generators; because of relation (i) a shortest expression for y has alternating x_{12}'s and x_{21}'s, say $y = x_{12}(a_1)\, x_{21}(b_1) \cdots x_{12}(a_t)\, x_{21}(b_t)$ with all a_i's and b_i's $\neq 0$. The case that we begin with an x_{21} or end with an x_{12} makes no difference. We may assume that $d(a_1) > 0$, since otherwise we can absorb $x_{12}(a_1)$ into the factor $m \in \mathrm{ST}(2, R_0)$, thus shortening the word for y. We let the matrix $\rho(y) = e_{12}(a_1) \cdots e_{21}(b_t)$ act as a product of transvections from the right on row vectors in R^2. Now $(1, 1)e_{12}(a_1) = (1, \beta_1) = (\alpha_1, \beta_1)$ with $\beta_1 = 1 + a_1$, hence $d(\beta_1) > 0$. Thus in the sequence of vectors (α_k, β_k) which result from the successive action of the transvections on $(1, 1) = (\alpha_0, \beta_0)$, at least some of the components have positive degree. Among the finitely many degrees occurring, there is a maximum, say D. Call the last vector (α_k, β_k) for which this maximum is reached (α, β). On the other hand, since $(1, 1)$ is the image of a vector in $R_0{}^2$ under the action of the matrix $\rho(m) \in \mathrm{EL}'(2, R_0)$ and since $\rho(m)\, p(y) = 1$, the vector $(1, 1)\, \rho(y)$ is in $R_0{}^2$. Therefore (α, β) cannot be the very last vector in the sequence and $d(\alpha) \neq d(\beta)$, since otherwise the degree D would still be reached in the next vector.

Assume for definiteness that $d(\alpha) < d(\beta) = D$. As the degree is lowered by the next transvection, this must be of the type $e_{12}(a_{s+1})$ for some $s < t$. Then $(\alpha, \beta) = (\alpha', \beta)\, e_{21}(b_s) = (\alpha' + \beta b_s, \beta)$ with $d(\alpha') \leqslant D$. Since $d(\beta b_s) \geqslant D$, this shows that $d(\alpha') = d(\beta) = D$ and $d(b_s) = 0$. In the preceding step, $(\alpha', \beta) = (\alpha', \beta')\, e_{12}(a_s)$ and we find that also $d(a_s) = 0$. As long as both components of the image vector have degree D, the intervening a_i's and b_i's all have degree 0. Working our way back step by step, we eventually encounter a vector (α'', β'') with $d(\alpha'') \neq d(\beta'')$.

We must distinguish two cases for this first vector with unequal degrees on our track back:

Case 1. $d(\beta'') < d(\alpha'') = D$.

Then in the next vector $(\alpha'', \beta'')\, e_{12}(a_r)$ both degrees equal D, and the product $e_{12}(a_r)\, e_{21}(b_r) \cdots e_{12}(a_s)\, e_{21}(b_s)$ is therefore a matrix $\left(\begin{smallmatrix} f & g \\ p & q \end{smallmatrix}\right)$ in $\mathrm{EL}'(2, R_0)$ whose action transforms (α'', β'') to (α, β). Note also that $r > 1$.

In the equations

$$\begin{aligned} \alpha &= \alpha'' f + \beta'' p \\ \beta &= \alpha'' g + \beta'' q \end{aligned} \tag{C}$$

the coefficients f, g, p, and q all have degrees $\leqslant 0$, while $d(\beta'') < d(\alpha'') = D = d(\beta) > d(\alpha)$. It follows that $f = 0$, whence g and p are units. From the matrix identity

$$\begin{pmatrix} 0 & g \\ p & q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ qg^{-1} & 1 \end{pmatrix} \begin{pmatrix} -gp & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -p^{-1} \\ p & 0 \end{pmatrix}$$

we conclude that the unit $u = -gp$ is in $V(2, R_0) =$ the commutator group $[R_0^{\bullet}, R_0^{\bullet}]$ in virtue of our assumption. If u is a product of elements $vwv^{-1}w^{-1}$, then a product of elements of type $h_{12}(vw)\, h_{12}(v)^{-1}h_{12}(w)^{-1}$ in $\mathrm{ST}'(2, R_0)$, say h, maps to $\mathrm{diag}(u, 1)$ under ρ. In $\mathrm{ST}'(2, R_0)$ we may therefore write $x_{12}(a_r) \cdots x_{21}(b_s) = x_{21}(qg^{-1})\, hw_{21}(p)z$ with some $z \in K_2(2, R_0)$.

Putting $c_{r-1} = b_{r-1} + qg^{-1}$, we see that in $\mathrm{ST}(2, R)$

$$x = m \cdot x_{12}(a_1) \cdots x_{12}(a_{r-1})\, x_{21}(c_{r-1})\, hw_{21}(p) \cdot z \cdot x_{12}(a_{s+1}) \cdots x_{21}(b_t) \cdot n.$$

We first shove the element z to the right past all the x_{ij}'s, in the process of which it gets conjugated a number of times, to finally become $z' \in N$, after which we write $n' = z'n$. We then push $hw_{21}(p)$ to the left past all the x_{ij}'s, which get flipped around in the process, according to the formulas recalled in Section 1. Writing $m' = mhw_{21}(p)$ in $\mathrm{ST}'(2, R_0)$ we obtain $x = m'y'n'$ where, assuming $x_{21}(c_{r-1})$ is flipped to $x_{12}(c'_{r-1})$ and putting $b'_{r-1} = c'_{r-1} + a_{s+1}$, we find

$$y' = x_{21}(a_1')\, x_{12}(b_1') \cdots x_{21}(a'_{r-1})\, x_{12}(b'_{r-1})\, x_{21}(b_{s+1}) \cdots x_{21}(b_t).$$

Since y' clearly is a shorter word in Steinberg generators than y, we have secured our induction step.

Case 2. $d(\alpha'') < d(\beta'') = D$.

Here the next step is of type $e_{21}(b_r)$ with $d(b_r) = 0$. Writing the product $e_{21}(b_r) \cdots e_{12}(a_s)\, e_{21}(b_s)$ as a matrix in $\mathrm{EL}'(2, R_0)$, the degree inequalities for Eqs. (C) now read $d(\alpha'') < d(\beta'') = D = d(\beta) > d(\alpha)$.

This time it transpires that $p = 0$ and hence that f and q are units. The matrix product

$$\begin{pmatrix} f & g \\ 0 & q \end{pmatrix} = \begin{pmatrix} 1 & gq^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} fq & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$$

shows that $fq \in [R_0^{\bullet}, R_0^{\bullet}]$. In $\mathrm{ST}'(2, R_0)$ therefore $x_{21}(b_r) \cdots x_{21}(b_s) = x_{21}(gq^{-1})\, hz$ for some h which is a product of $h_{ij}(u)$'s, $u \in R_0^{\bullet}$, and a $z \in K_2'(2, R_0)$. Amalgamating $x_{12}(a_r)$ with $x_{12}(gq^{-1})$ to $x_{12}(c_r)$, then shoving h to the left and z to the right, we obtain a word

$$y' = x_{12}(a_1')\, x_{21}(b_1') \cdots x_{21}(b'_{r-1})\, x_{12}(a'_{s+1})\, x_{21}(b_{s+1}) \cdots x_{21}(b_t)$$

in which we have further merged the terms $x_{12}(c_r')$ and $x_{12}(a_{s+1})$ to $x_{12}(a'_{s+1})$. Again y' is shorter than y and we have achieved our aim.

The following consequence extends Bak's original result, which dealt with polynomials over a field.

THEOREM 10. *Let k be an integral domain, not necessarily commutative, which is universal for* GE_2. *Then* $K_2(2, k[t_1, \ldots, t_m]) = K_2(2, k)$ and $K_2(2, k\langle X\rangle) =$

$K_2(2, k)$ for any finite or infinite number m of polynomial indeterminates and any set X.

Proof. Since k has no zero divisors, the total degree determines an integer valued degree function on the polynomial ring $k[t_1, \ldots, t_m]$. In the free algebra $k\langle X\rangle$ one defines the degree of a polynomial in noncommuting indeterminates as the length of its longest monomial [3, p. 32]. Calling both these rings R, we have $R_0 = k$. Moreover both rings are augmented, which allows us to drop the dash from $K_2'(2, k)$ and the other groups. Because k is universal for GE_2, we know from Theorem 5 that $K_2(2, k) \subset H(2, k)$. The latter is contained in $H(2, R)$, in fact equals it, because the bigger ring contains no additional units. Hence $K(2, k)$ is central in $\mathrm{ST}(2, R)$. Finally $V(2, k) = [k^\bullet, k^\bullet]$ again by Theorem 5, so we may invoke Theorem 9 to obtain the result.

In particular, a field being universal for GE_2, the above answers afirmatively [7, Problem 8] in the case $n = 2$.

THEOREM 11. *If k is a field, possibly skew, and X any set, then $K_2(n, k\langle X\rangle) = K_2(n, k)$, $n \geqslant 2$.*

Proof. For $n \geqslant 3$, use Silvester's result [14, Theorem 6], that $k\langle X\rangle$ is universal for GE_n. By Theorem 5 this implies that $K_2(n, k\langle X\rangle) = C(n, k\langle X\rangle)$. The latter equals $C(n, k) = K_2(n, k)$ because all units are in k. The result for $n = 2$ is a special case of the previous theorem.

Whether $K_2(n, -)$, $n \geqslant 3$, remains unchanged under polynomial extensions I do not know.

Finally we quote an important result due to Cohn [3, Theorem 3.4; 1, IV, Corollary 5.17] which asserts that $\mathrm{GL}(n, k\langle X\rangle)$ is generated by diagonal and elementary matrices:

THEOREM 12. *Let k be a field, possibly skew, and X any set. Then* $\mathrm{GL}(n, k\langle X\rangle) = \mathrm{GE}(n, k\langle X\rangle)$ *for $n \geqslant 1$.*

Remark. In these and the following results, it is immaterial whether we conceive of X as a set with or without base point.

6. IDENTIFYING THE UNIVERSAL COVERING

In this task, we freely resort to notations introduced and facts discussed in Sections 2 and 4.

THEOREM 13. *For any field k and any set X*

$$\mathrm{GL}(n, \langle X\rangle) = \mathrm{EL}(n, \langle X\rangle) = \mathrm{ST}(n, \langle X\rangle).$$

Proof. In the commuting square of surjective homomorphisms

$$\begin{tikzcd} \mathrm{ST}(n, k\langle X\rangle) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{EL}(n, k\langle X\rangle) \arrow[d, two heads] \\ \mathrm{ST}(n, k) \arrow[r, two heads] & \mathrm{EL}(n, k) \end{tikzcd}$$

the vertical kernels $\mathrm{ST}(n, \langle X\rangle)$ and $\mathrm{EL}(n, \langle X\rangle)$ are isomorphic precisely when the horizontal kernels $K_2(n, k\langle X\rangle)$ and $K_2(n, k)$ are. The latter are isomorphic by Theorem 11.

To prove the first identity, use that every $M \in \mathrm{GL}(n, k\langle X\rangle)$ is a product of elementary and diagonal matrices $d(u_1, \ldots, u_n)$ with $u_i \in k\langle X\rangle^{\bullet} = k^{\bullet}$ according to Theorem 12. Applying relation (v)′, push all the diagonals to the right and amalgamate them using (iv)′. Write the occurring $p \in k\langle X\rangle$ as $p = c + q$, $c \in k$, $q \in \langle X\rangle$, and employ (i)′ to obtain

$$M = \prod_{\lambda=1}^{m} e_{ij}^{(\lambda)}(c_\lambda)\, e_{ij}^{(\lambda)}(q_\lambda) \cdot d(v_1, \ldots, v_n), \qquad v_i \in k^{\bullet},$$

where $e_{ij}^{(\lambda)}$ is short for $e_{i_\lambda j_\lambda}$. If M is actually in $\mathrm{GL}(n, \langle X\rangle)$, then $d(v_1, \ldots, v_n)^{-1} = \prod_{\lambda=1}^{m} e_{ij}^{(\lambda)}(c_\lambda) \in \mathrm{EL}(n, k)$, hence $M \in \mathrm{EL}(n, \langle X\rangle)$. As we have seen in Section 3, Example 3, M is a product of conjugates under $\mathrm{EL}(n, k)$ of elements $e_{ij}(q)$, $q \in \langle X\rangle$.

From now on A denotes an algebra over a field k.

COROLLARY 14. $\mathrm{GL}(n, EA) = \mathrm{EL}(n, EA) = \mathrm{ST}(n, EA)$.

We want to determine the group $\mathrm{GL}(n, A)$ by applying diagram (B) of Section 4 with $F = \mathrm{GL}_n$ to the algebra A, and to this purpose introduce some more notation. The functor $T\colon \mathrm{Alg}_k \to \mathrm{Alg}_{k1}$ of Section 2 is for convenience indicated by the subscript k. Following [21, Sections 1 and 8], write $\Omega A = \mathrm{Ker}(\epsilon(A)\colon EA \to A)$ and $C_2A = \mathrm{Ker}(E\epsilon(A)\colon E^2A \to EA)$. Notice that the homomorphism $\epsilon E(A)$ restricted to C_2A maps this algebra into ΩA since $\epsilon \circ \epsilon E = \epsilon \circ E\epsilon$. It also maps onto because $E\Omega A \subset C_2A$ is already mapped onto ΩA by $\epsilon E(A)$. A further observation is that ΩA is also $\mathrm{Ker}(\epsilon_k(A)\colon k\langle A\rangle \to A_k)$ and $C_2A = \mathrm{Ker}((E\epsilon)_k\colon k\langle EA\rangle \to k\langle A\rangle)$.

With this notation fixed, consider the commuting diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & C_2A \arrow[r, phantom, "="] \arrow[d] & C_2A \arrow[d] & & \\
0 \arrow[r] & E^2A \arrow[r] \arrow[d, "E\epsilon(A)"'] & k\langle EA\rangle \arrow[r, "\tau"] \arrow[d, "(E\epsilon)_k(A)"] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & EA \arrow[r] \arrow[d] & k\langle A\rangle \arrow[r, "\tau"'] \arrow[d] & k \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

in Alg_k with exact rows and columns. The homomorphism $\mu(A)$ splits the left column, so $\mu_k(A)$ splits the middle column; also the augmentations τ are split. From the left exactness of GL_n it follows that the diagram remains exact after applying this functor, so $\mathrm{GL}(n, C_2A)$ is the group $M_2(A)$ of diagram (B). On the other hand, application of ST_n keeps at least the rows exact because $\mathrm{ST}(n, E^2A)$ and $\mathrm{ST}(n, EA)$ are defined as the appropriate kernels. Using Corollary 14, we see that $\mathrm{GL}(n, C_2A)$ may be identified with $\mathrm{Ker}\, \mathrm{ST}(n, E\epsilon(A))$ and then with $\mathrm{Ker}\, \mathrm{ST}(n, (E\epsilon)_k(A))$. Hence $\mathrm{GL}(n, C_2A)$ is the normal subgroup of $\mathrm{ST}(n, k\langle EA\rangle)$ generated by all elements $x_{ij}(y)$, $y \in C_2A$, under conjugation by $\mathrm{ST}(n, k\langle A\rangle)$, the latter being embedded in $\mathrm{ST}(n, k\langle EA\rangle)$ by the monomorphism $\mathrm{ST}(n, \mu_k\langle A\rangle)$.

Diagram (B) now reads

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{GL}(n, C_2A) \arrow[r] \arrow[d, "d(A)"'] & \mathrm{GL}(n, E^2A) \arrow[r, "\mathrm{GL}(n, E\epsilon(A))"] \arrow[d, "\mathrm{GL}(n, \epsilon E(A))"'] & \mathrm{GL}(n, EA) \arrow[r] \arrow[d, "\mathrm{GL}(n, \epsilon(A))"] & 1 \\
1 \arrow[r] & \mathrm{GL}(n, \Omega A) \arrow[r] & \mathrm{GL}(n, EA) \arrow[r, "\mathrm{GL}(n, \epsilon(A))"'] & \mathrm{GL}(n, A) &
\end{tikzcd}$$

In view of Corollary 14, the middle column is the map $\mathrm{ST}(n, \epsilon E(A))$ and we determine $\mathrm{Im}\, d(A)$ in $\mathrm{GL}(n, EA) = \mathrm{ST}(n, EA)$ which is the normal subgroup of $\mathrm{ST}(n, k\langle A\rangle)$ generated by all $x_{ij}(q)$, $q \in EA$, under conjugation by $\mathrm{ST}(n, k)$. This image then is the normal subgroup of $\mathrm{ST}(n, k\langle A\rangle)$ generated by all $x_{ij}(\omega)$, $\omega \in \Omega A$, under conjugation by $\mathrm{ST}(n, k\langle A\rangle)$. Since we may also conjugate the $x_{ij}(q)$'s with the whole of $\mathrm{ST}(n, k\langle A\rangle)$, we have proved

PROPOSITION 15. $\widehat{\mathrm{GL}}(n, A) = \mathrm{ST}(n, EA)/\mathrm{Im}\, d(A)$ *is the quotient of two normal subgroups of* $\mathrm{ST}(n, k\langle A\rangle)$; *the bigger one normally generated by all* $x_{ij}(q)$, $q \in EA$, *the smaller one by all* $x_{ij}(\omega)$, $\omega \in \Omega A$.

Now consider the homomorphism $\epsilon_k(A)\colon k\langle A\rangle \to A_k$ of augmented k-algebras. Since $\epsilon(A)\colon EA \to A$ is its restriction to the augmentation ideals, both morphisms have ΩA as their kernel. Applying the functor ST_n to these maps, we find $\mathrm{Ker}(\mathrm{ST}(n, k\langle A\rangle) \to \mathrm{ST}(n, A_k)) = \mathrm{Ker}(\mathrm{ST}(n, EA) \to \mathrm{ST}(n, A_k, A))$. The former is by definition $\mathrm{ST}(n, k\langle A\rangle, \Omega A)$ and for $n \geqslant 3$ this is the normal subgroup N of $\mathrm{ST}(n, k\langle A\rangle)$ generated by all the conjugates of the $x_{ij}(\omega)$, $\omega \in \Omega A$. Thus we have established that $\widehat{\mathrm{GL}}(n, A) \simeq \mathrm{ST}(n, A_k, A)$ for $n \geqslant 3$.

In the case $n = 2$, however, we observe that $k\langle A\rangle^\bullet = k^\bullet$ and $\epsilon_k(k^\bullet)$ consists of those units in A_k which have the form $(0, u)$, $u \in k^\bullet$. Our observations in Section 2 then give us the following presentation for the group $\mathrm{ST}(2, k\langle A\rangle)/N$: it is generated by all elements $\bar{x}_{ij}((a, u))$ (harmlessly written $\bar{x}_{ij}(a, u)$) subject to the relations i) and

$$^{\bar{w}_{ij}(0,v)}\bar{x}_{ji}(a, u) = \bar{x}_{ij}(-vav, -vuv) \qquad \text{for} \quad (a, u) \in A_k, \quad v \in k^\bullet;$$

since k is in the center of A_k the latter element equals $\bar{x}_{ij}(-v^2a, -v^2u)$. If A has an identity element e, the units of A_k are of the form $(0, u)(a, 1)$ where a can be any element in A with $a + e \in A^\bullet$ and $u \in k^\bullet$, so that A_k as a rule contains (many) more units than $k^\bullet$. Therefore the group $\mathrm{ST}(2, k\langle A\rangle)/N$ has to be distinguished from $\mathrm{ST}(2, A_k)$ and we denote it by $\overline{\mathrm{ST}(2, A_k)}$. Sending $\bar{x}_{ij}(a, u)$ to $x_{ij}(u)$ induces a homomorphism $\overline{\mathrm{ST}(2, A_k)} \to \mathrm{ST}(2, k)$. This epimorphism admits a section given by $x_{ij}(u) \mapsto \bar{x}_{ij}(0, u)$ and therefore its kernel $\overline{\mathrm{ST}(2, A_k, A)}$ is the normal subgroup of $\overline{\mathrm{ST}(2, A_k)}$ consisting of all elements $^{\bar{x}_{ji}(0,v)}\bar{x}_{ij}(a, 0)$, $a \in A$, $v \in k^\bullet$, and their products. Now $\mathrm{ST}(2, EA)/N \simeq \overline{\mathrm{ST}(2, A_k, A)}$, which means that the universal covering $\widehat{\mathrm{GL}}(2, A)$ is isomorphic to this last group.

There is a short exact sequence of groups

$$1 \to K_2(n, A_k, A) \to \mathrm{ST}(n, A_k, A) \to \mathrm{EL}(n, A_k, A) \to 1$$

induced by $\rho\colon \mathrm{ST}(n, A_k) \to \mathrm{EL}(n, A_k)$. We further define $\overline{K(2, A_k)} = \mathrm{Ker}(\overline{\mathrm{ST}(2, A_k)} \to \mathrm{EL}(2, A_k))$ and similarly the relative group $\overline{K_2(2, A_k, A)}$ fits into a short exact sequence.

From now on, we assume that $A \in \mathrm{Alg}_{k1}$. Recalling the isomorphism $A_k \simeq A \times k$ of augmented unital k-algebras given by $(a, u) \mapsto (a + ue) \times u$ we find that $\mathrm{EL}(n, A_k, A) \simeq \mathrm{EL}(n, A \times k, A)$. However, the latter may be identified with $\mathrm{EL}(n, A)$ since EL_n preserves products. On the other hand, the universal covering $\Lambda\colon \widehat{\mathrm{GL}}(n, A) \to \mathrm{GL}(n, A)$ is induced from the composite map $\mathrm{GL}(n, EA) = \mathrm{ST}(n, EA) \to \mathrm{ST}(n, A) \to \mathrm{EL}(n, A) \to \mathrm{GL}(n, A)$ in which $x_{ij}(q)$ is sent to $e_{ij}(\epsilon(q))$.

Bringing the defining sequence (A) of the homotopy groups into play, we have said enough to justify

THEOREM 16. *For a unital algebra A over a field k and $n \geqslant 3$, there is a* *functorial isomorphism between exact sequences of groups (except for* "GL$(n, A)/$EL(n, A)" *which is in general only a homogeneous space of cosets):*

$$\begin{array}{ccccccccc}
1 \to & \pi_1\mathrm{GL}(n, A) & \to & \widehat{\mathrm{GL}}(n, A) & \to & \mathrm{GL}(n, A) & \to & \pi_0\mathrm{GL}(n, A) & \to 1 \\
& \downarrow \wr & & \downarrow \wr & & \| & & \downarrow \wr & \\
1 \to & K_2(n, A_k, A) & \to & \mathrm{ST}(n, A_k, A) & \to & \mathrm{GL}(n, A) & \to & \mathrm{GL}(n, A)/\mathrm{EL}(n, A) & \to 1
\end{array}$$

In the case $n = 2$, the groups ST$(2, A_k, A)$ *and* $K_2(2, A_k, A)$ *in the bottom sequence should carry a bar. In particular, the connected component of* GL(n, A) *is* EL(n, A) *for all n.*

7. Presenting the Universal Covering

For $A \in \mathrm{Alg}_{k1}$, k a field, and $n \geqslant 3$, we have identified $\widehat{\mathrm{GL}}(n, A)$ with ST(n, A_k, A) and would like to know when the latter equals ST(n, A). Recall the isomorphism $A_k \xrightarrow{\sim} A \times k$ of augmented k-algebras. From the commuting diagram of group homomorphisms

$$\begin{tikzcd}
\mathrm{ST}(n, A_k) \arrow[r, "\sim"] \arrow[dr] & \mathrm{ST}(n, A \times k) \arrow[r, two heads] \arrow[d] & \mathrm{ST}(n, A) \times \mathrm{ST}(n, k) \arrow[dl] \\
& \mathrm{ST}(n, k) &
\end{tikzcd}$$

we see that ST(n, A_k, A) is isomorphic to ST(n, A) if and only if ST$(n, A \times k) \simeq$ ST$(n, A) \times$ ST(n, k). Since EL$_n$ preserves products, this is equivalent to the condition that $K_2(n, A \times k) \simeq K_2(n, A) \times K_2(n, k)$.

The following proposition is due to Stein [18, Lemma 2.12], but we sketch a proof to show the idea involved.

Proposition 17. *Let R and S be unital rings and let $T_n \lhd$ ST$(n, R \times S)$ be the normal subgroup generated by all elements $\langle (r, 0), (0, s)\rangle_{ij}$. Then* Ker(ST$(n, R \times S) \to$ ST$(n, R) \times$ ST$(n, S)) = T_n$ *for all n and vanishes for $n \geqslant 3$.*

Proof. The mapping being given by $x_{ij}(r, 0) \mapsto x_{ij}(r) \times 1$, $x_{kl}(0, s) \mapsto 1 \times x_{kl}(s)$ on the generators, and since these images commute in ST$(n, R) \times$ ST(n, S), the kernel must certainly contain all $[x_{ij}(r, 0), x_{kl}(0, s)]$, $r \in R$, $s \in S$ and their conjugates. In virtue of relations (i) and (ii) in ST$(n, R \times S)$, these commutators are 1 unless $k = j$ and $i = l$. The kernel therefore contains T_n, normally generated by the $\langle (r, 0), (0, s)\rangle_{ij} = [x_{ij}(r, 0), x_{ji}(0, -s)]^{-1}$. In case $n \geqslant 3$, $x_{ji}(0, -s) = [x_{jk}(0, -s), x_{ki}(0, 1)]$ for some $k \neq i$ or j, hence $\langle (r, 0), (0, s)\rangle_{ij} = 1$. One now verifies that ST$(n, R \times S)/T_n$ has a presentation in terms of the commuting images of the $x_{ij}(r, 0)$ and $x_{kl}(0, s)$ which is identical to that of $\mathrm{ST}(n, R) \times \mathrm{ST}(n, S)$ in terms of the generators $x_{ij}(r) \times 1$ and $1 \times x_{kl}(s)$.

PROPOSITION 18. *If A and B are unital semilocal algebras over a field $k \neq \mathbf{F}_2$, $\mathbf{F}_3$, then* $\mathrm{ST}(2, A \times B) \simeq \mathrm{ST}(2, A) \times \mathrm{ST}(2, B)$.

Proof. We prove that the obstructions $\langle (a, 0), (0, b) \rangle_{ij}$ all vanish. First we claim that every $a \in A$ can be written as a sum $a = \sum_l (u_l - 1)$ with $u_l \in A^{\bullet}$. It is enough to prove this modulo the radical of A, say J. Then $\bar{A} = A/J$ is a finite product of full matrix algebras $M_{m_l}(D_l)$ over skew fields D_l which are extensions of k. Again, it is enough to prove that $a = \sum_l (u_l - 1)$ is true for every a in each such component, since we can add terms $(1 - 1)$ ad lib. In the simple matrix algebra $M_m(D)$ we write $e_{ii}(v)$ for the diagonal matrix which only differs from the identity matrix in the i, ith spot where the entry is $1 + v$, v in the skew field D.

Now an arbitrary matrix (α_{ij}) in $M_m(D)$ can be written as $(\alpha_{ij}) = \sum_{i,j=1}^{m} (e_{ij}(\alpha_{ij}) - 1)$. The only noninvertible matrices which may possibly occur are $e_{ii}(-1)$. But since $\mathbf{F}_2 \neq k \subset D$, we can write $-1 = v + w$ with $v, w \neq -1$ units in k and replace the term $e_{ii}(-1) - 1$ by $(e_{ii}(v) - 1) + (e_{ii}(w) - 1)$, gaining our objective.

Now if $a = u - 1$, $b = v - 1$, with $u \in A^{\bullet}$, $b \in B^{\bullet}$, then $\langle (a, 0), (0, b) \rangle_{ij} = \{(u, 1), (1, v)\}_{ji}$ by Lemma 2, and this is a central element in $\mathrm{ST}(2, A \times B)$. The commutator formula $[\alpha\beta, \gamma] = {}^{\alpha}[\beta, \gamma][\alpha, \gamma]$ and relation (i) permit one to show by induction that if $a = \sum_l (u_l - 1)$, $u_l \in A^{\bullet}$, and b as before, then $\langle (a, 0), (0, b) \rangle_{ij} = \prod_l \{(u_l, 1), (1, v)\}_{ji}$ which is again central. The first part of the proof and a similar argument for b then yield that the symbol $\langle (a, 0 \rangle, (0, b) \rangle_{ij}$ is a central function which is additive in a and b (although of course we continue to write $\mathrm{ST}(2, A \times B)$ multiplicatively).

Next conjugate the central element $\langle (a, 0), (0, c) \rangle_{ij} = [x_{ij}(a, 0), x_{ji}(0, -c)]^{-1}$ by $h_{ji}(1, v)$ with $v \in B^{\bullet}$ to obtain $\langle (a, 0), (0, c) \rangle_{ij} = \langle (a, 0), (0, vcv) \rangle_{ij}$, applying the relation mentioned in Section 1. Since k is in the center of B, and we have excluded the two smallest fields, we may find a central unit v in k with $v^2 - 1 \in k^{\bullet}$ and consequently $b = (v^2 - 1)c$ for a certain $c \in B$. Hence $\langle (a, 0), (0, b) \rangle_{ij} = 1$.

The case $k = \mathbf{F}_2$ is a genuine exception, since it is known that $\mathrm{ST}(2, \mathbf{F}_2 \times \mathbf{F}_2) \not\simeq \mathrm{ST}(2, \mathbf{F}_2) \times ST(2, \mathbf{F}_2) = 1$ [8, Appendix]. For $\mathbf{F}_3$ I can prove the proposition for certain types of algebras (see for instance the following corollary) but I do not know the full story. It is clear that the method also yields results for semilocal rings, but investigating the exceptions is a more delicate matter.

COROLLARY 19. $\mathrm{ST}(n, A_k, A) \simeq \mathrm{ST}(n, A)$ *and* $K_2(n, A_k, A) \simeq K_2(n, A)$ *for every unital algebra A over a field k if $n \geqslant 3$; for semilocal algebras if $n = 2$ and $k \neq \mathbf{F}_2$.*

Proof. In view of the above, it remains only to prove the previous proposition when $k = B = \mathbf{F}_3$. To this purpose, use the identity $\{u, v\}_{ij} = \{u, -uv\}_{ij} =$

$\{-uv, v\}_{ij}$ which holds whenever u and v are commuting units in a ring [18, Proposition 1.1, (S4)]. Assume $1 + a \in A^{\bullet}$ for a certain $a \in A$. Then $\langle(a, 0), (0,1)\rangle_{ij} = \{(1 + a, 1), (1, -1)\}_{ji} = \{(-1 - a, 1), (1, -1)\}_{ji} = \langle(1 - a, 0), (0, 1)\rangle_{ij}$. Hence $\langle(a, 0), (0, 1)\rangle_{ij}^2 = \langle(1, 0), (0, 1)\rangle_{ij}$ for every such a. In particular $\langle(1, 0), (0, 1)\rangle_{ij} = 1$. An element $\langle(a, 0), (0, 1)\rangle_{ij}$ thus has order 2, but since every element has order 3 because of the characteristic, it must be 1. By additivity and the first part of the proof of Proposition 18, it follows that $\langle(0, a), (0, b)\rangle_{ij} = 1$ for every $a \in A$, $b \in k$, which ends the proof.

If A is any unital algebra over a field k, identifying $\widehat{\mathrm{GL}}(n, A)$ with $\mathrm{ST}(n, A_k, A) \simeq \mathrm{ST}(n, A)$ for $n \geqslant 3$, we know that $\widehat{\mathrm{GL}}(n, A)$ has the Steinberg presentation with relations (i) and (ii). On the other hand, the group $\widehat{\mathrm{GL}}(2, A) \simeq \overline{\mathrm{ST}(2, A_k, A)}$ was described as the normal subgroup of $\overline{\mathrm{ST}(2, A_k)}$ consisting of all elements ${}^{\bar{x}_{ji}(0,v)}\bar{x}_{ij}(a, 0)$ and their products. We now seek a presentation for this group.

Consider the set $k \cup \infty$ in which the symbol ∞ satisfies the usual rules of addition and multiplication with regard to k. We call this set $\mathbf{P}_k^1$, loosely suggesting the projective line over k. Let A^+ be the additive group of the algebra A, and let Y stand for the free product of copies of A^+, indexed by $\mathbf{P}_k^1$. We write this group multiplicatively, thus Y has generators $y(a, u)$, $a \in A$, $u \in \mathbf{P}_k^1$ satisfying the relations $y(0, u) = 1$; $y(a, u)\, y(b, u) = y(a + b, u)$ for all $u \in \mathbf{P}_k^1$, $a, b \in A$. An element of Y is then a word in these generators which can only be shortened through these relations. We now propose to define an action of $\mathrm{ST}(2, k)$ on Y by putting

$$
\begin{aligned}
x_{12}(v) * y(a, u) &= y(a, u + v) && \text{for all } v \in k;\\
x_{21}(v) * y(a, u) &= y((uv + 1)^2 a, u/(uv + 1)) && \text{if } u \in k,\ uv \neq -1\\
&= y(-u^2 a, \infty) && \text{if } u \in k,\ uv = -1\\
&= y(-v^2 a, 1/v) && \text{if } u = \infty \text{ and } v \neq 0\\
&= y(a, \infty) && \text{if } u = \infty \text{ and } v = 0.
\end{aligned}
$$

These definitions are seen to entail, for every $z \in k^{\bullet}$,

$$
\begin{aligned}
w_{12}(z) * y(a, u) &= y((u/z)^2 a, -z^2/u)) && \text{if } u \neq 0, \infty\\
&= y(-z^2 a, \infty) && \text{if } u = 0\\
&= y(-(1/z)^2 a, 0) && \text{if } u = \infty.
\end{aligned}
$$

First remark that all formulas are additive in a, hence action by an element of $\mathrm{ST}(2, k)$ does indeed yield an automorphism of the group Y. It is clear that this action respects the relation $x_{12}(z)\, x_{12}(v) = x_{12}(z + v)$ in $\mathrm{ST}(2, k)$. Conjugating this relation with $w_{12}(1)$ produces $x_{21}(-z)\, x_{21}(-v) = x_{21}(-z - v)$, and since $w_{21}(z) = w_{12}(-z^{-1})$ for $z \in k^{\bullet}$, it therefore is enough to check that

${}^{w_{12}(z)}x_{21}(v) = x_{12}(-z^2v)$ is carried over by the action. A straightforward calculation shows that

$$\begin{aligned} {}^{w_{12}(z)}x_{21}(v) * y(a, u) &= w_{12}(z)\, x_{21}(v) * y((u/z)^2 a, -(z^2/u)) \\ &= w_{12}(z) * y(((u - z^2v)/z)^2 a, z^2(z^2v - u)^{-1}) \\ &= y(a, u - z^2v) = x_{12}(-z^2v) * y(a, u) \end{aligned}$$

for $u \neq 0, \infty$, $v \in k$, $a \in A$, and $u \neq z^2v$. Verification of all the exceptional cases is left to the patient reader. Relations (i) and (iii) are thus seen to be fully preserved and we have shown that Y is a group upon which $\mathrm{ST}(2, k)$ operates in the fashion described.

We now construct a semidirect product $Z = Y * \mathrm{ST}(2, k)$ by defining multiplication in Z as $y_1x_1 \cdot y_2x_2 = y_1(x_1 * y_2) \cdot x_1x_2$, $y_i \in Y$, $x_i \in \mathrm{ST}(2, k)$.

PROPOSITION 20. *There exists an isomorphism of split group extensions*

$$\begin{tikzcd} 1 \arrow[r] & Y \arrow[r] \arrow[d, "\wr"] & Z \arrow[r] \arrow[d, "\vartheta"', "\wr"] & \mathrm{ST}(2, k) \arrow[r] \arrow[d, "\wr"] & 1 \\ 1 \arrow[r] & \overline{\mathrm{ST}(2, A_k, A)} \arrow[r] & \overline{\mathrm{ST}(2, A_k)} \arrow[r] & \mathrm{ST}(2, k) \arrow[r] & 1 \end{tikzcd}$$

given by

$$\vartheta(x_{ij}(u)) = \bar{x}_{ij}(0, u), \qquad \vartheta(y(a, u)) = {}^{\bar{x}_{12}(0,u)}\bar{x}_{21}(a, 0),$$
$$\vartheta(y(a, \infty)) = \bar{x}_{12}(a, 0), \qquad a \in A,\ u \in k,$$

and

$$\vartheta(x * y) = {}^{\vartheta(x)}\vartheta(y), \qquad x \in \mathrm{ST}(2, k),\ y \in Y.$$

Proof. Clearly the relations $y(0, u) = 1$ and $y(a + b, u) = y(a, u)\, y(b, u)$ are preserved by ϑ, and $\mathrm{ST}(2, k)$ is taken isomorphically to its copy in $\overline{\mathrm{ST}(2, A_k)}$. Let us check that the operation $*$ of $\mathrm{ST}(2, k)$ on Y is carried under ϑ to conjugation by $\mathrm{ST}(2, k)$ in $\overline{\mathrm{ST}(2, A_k, A)}$. It is enough to do this for the generators. Well,

$$\begin{aligned} \vartheta(x_{12}(v) * y(a, u)) &= \vartheta(y(a, u + v)) = {}^{\bar{x}_{12}(0,u+v)}\bar{x}_{21}(a, 0) \\ &= {}^{\bar{x}_{12}(0,u)\bar{x}_{12}(0,v)}\bar{x}_{21}(a, 0) \\ &= {}^{\bar{x}_{12}(0,v)}\vartheta(y(a, u)) \qquad \text{for} \quad u, v \in k, \quad a \in A, \end{aligned}$$

and

$$\begin{aligned} \vartheta(x_{12}(v) * y(a, \infty)) &= \vartheta(y(a, \infty)) = \bar{x}_{12}(a, 0) \\ &= {}^{\bar{x}_{12}(0,v)}\bar{x}_{12}(a, 0) = {}^{\bar{x}_{12}(0,v)}\vartheta(y(a, \infty)). \end{aligned}$$

Next the calculation

$$\begin{aligned}
\vartheta(x_{21}(v) * y(a, u)) &= \vartheta(y(uv+1)^2a, u/(uv+1)) = {}^{\bar{x}_{12}(0,u/(uv+1))}\bar{x}_{21}(uv+1)^2a, 0) \\
&= {}^{\bar{x}_{21}(0,(uv+1)/u)\bar{w}_{21}(0,-(uv+1)/u)}\bar{x}_{21}((uv+1)^2a, 0) \\
&= {}^{\bar{x}_{21}(0,v)\bar{x}_{21}(0,u^{-1})}\bar{x}_{12}(-u^2a, 0) \\
&= {}^{\bar{x}_{21}(0,v)\bar{x}_{12}(0,u)\bar{w}_{12}(0,-u)}\bar{x}_{12}(-u^2a, 0) \\
&= {}^{\bar{x}_{21}(0,v)\bar{x}_{12}(0,u)}\bar{x}_{21}(a, 0) = {}^{\bar{x}_{21}(0,v)}\vartheta(y(a, u)),
\end{aligned}$$

if $u \in k$, $uv \neq -1$, takes care of the general case, while if $uv = -1$ we have

$$\begin{aligned}
\vartheta(x_{21}(v) * y(a, u)) &= \vartheta(y(-u^2a, \infty)) = \bar{x}_{12}(-u^2a, 0) = {}^{\bar{w}_{21}(0,-u^{-1})}\bar{x}_{21}(a, 0) \\
&= {}^{\bar{x}_{21}(0,-u^{-1})\bar{x}_{12}(0,u)}\bar{x}_{21}(a, 0) = {}^{\bar{x}_{21}(0,v)}\vartheta(y(a, u)).
\end{aligned}$$

The remaining cases are disposed of similarly, so that ϑ is a homomorphism of groups and in fact of split extensions as required. All the generators $\bar{x}_{ij}(a, u) = \bar{x}_{ij}(a, 0)\,\bar{x}_{ij}(0, u)$ of $\overline{\mathrm{ST}(2, A_k)}$ are then in the image, so that ϑ is onto.

We construct an inverse to ϑ. Put $\eta(\bar{x}_{12}(a, u)) = y(a, \infty)\,x_{12}(u)$ and $\eta(\bar{x}_{21}(a, u)) = y(a, 0)\,x_{21}(u)$, $a \in A$, $u \in k$, and accordingly for products in these generators. It remains to verify that this defines a homomorphism $\eta\colon \overline{\mathrm{ST}(2, A_k)} \to Z$ by checking that the relations which hold in the former are preserved. To this purpose write down

$$\begin{aligned}
\eta(\bar{x}_{12}(a, u)) \cdot \eta(\bar{x}_{12}(b, v)) &= y(a, \infty)\,x_{12}(u) \cdot y(b, \infty)\,x_{12}(v) \\
&= y(a, \infty)(x_{12}(u) * y(b, \infty)) \cdot x_{12}(u)\,x_{12}(v) \\
&= y(a, \infty)\,y(b, \infty) \cdot x_{12}(u)\,x_{12}(v) \\
&= y(a+b, \infty)\,x_{12}(u+v) \\
&= \eta(\bar{x}_{12}(a+b, u+v)), \qquad a, b \in A,\ u, v \in k.
\end{aligned}$$

As in the case of $\mathrm{ST}(2, k)$ argued before, it then suffices to check that the relation ${}^{\bar{w}_{12}(0,z)}\bar{x}_{21}(a, u) = \bar{x}_{12}(-z^2a, -z^2u)$ is respected by η, $z \in k^{\bullet}$. Since $\eta(\bar{w}_{12}(0, z)) = w_{12}(z)$, we have $\eta({}^{\bar{w}_{12}(0,\alpha)}\bar{x}_{21}(a, u)) = {}^{w_{12}(z)}y(a, 0)^{w_{12}(z)}x_{21}(u) = y(-z^2a, \infty)\,x_{12}(-z^2u) = \eta(\bar{x}_{12}(-z^2a, -z^2u))$.
Clearly η is the inverse of the homomorphism ϑ, so the proposition is proved.

Combining this with Theorem 16 and Corollary 19 we have obtained a presentation of the universal covering in all cases:

THEOREM 21. *For every unital algebra A over a field k, $\widehat{\mathrm{GL}}(n, A) \simeq \mathrm{ST}(n, A)$ and is accordingly presented when $n \geqslant 3$. Furthermore, $\widehat{\mathrm{GL}}(2, A)$ is a free product of* card $\mathbf{P}_k^1$ *copies of A^+.*

8. The Fundamental Group

If A is a unital algebra over a field k, Theorem 16 and Corollary 19 tell us that $\pi_1 \mathrm{GL}(n, A) \simeq K_2(n, A)$ for $n \geqslant 3$. To discuss the case $n = 2$, recall the group Y of Proposition 20 and call it $Y(A)$ to emphasize functoriality. Consider the sequence of homomorphisms

$$Y(A) \longrightarrow \overline{\mathrm{ST}(2, A_k, A)} \longrightarrow \mathrm{ST}(2, A_k, A) \longrightarrow \mathrm{ST}(2, A)$$

which are given by $y(a, u) \mapsto \vartheta(y(a, u))$, ${}^{\bar{x}_{ij}(0,u)}\bar{x}_{ji}(a, 0) \mapsto {}^{x_{ij}(0,u)}x_{ji}(a, 0)$ and ${}^{x_{ij}(0,u)}x_{ji}(a, 0) \mapsto {}^{x_{ij}(u)}x_{ji}(a)$, respectively. Proposition 20 states that the first map is an isomorphism and Corollary 19 that the third is, if A is semilocal and $k \neq \mathbf{F}_2$. Write $Q(A)$ for the kernel of the composite homomorphism. There is a commutative diagram of group homomorphisms

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & Q(A) \arrow[r] \arrow[d, equal] & \overline{K_2(2, A_k, A)} \arrow[r] \arrow[d] & K_2(2, A) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Q(A) \arrow[r] & Y(A) \arrow[r] \arrow[d] & \mathrm{ST}(2, A) \arrow[r] \arrow[d] & 1 \\
 & & \mathrm{EL}(2, A) \arrow[r, equal] \arrow[d] & \mathrm{EL}(2, A) \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}$$

with exact rows and columns. Consider the elements

$$p(a, u) = y(a, u)\, y(u, \infty)\, y(-a, 0)\, y(-u, \infty)$$

and

$$q(a, c) = {}^{y(c,\infty)y(-c^{-1}0,)y(c,\infty)}y(a, 0) \cdot y(cac, \infty) \qquad \text{in } Y(A),\ a \in A,\ c \in A^{\bullet},\ u \in k.$$

They are clearly in $Q(A)$, so if $P(A)$ is the normal subgroup generated by them in $Y(A)$, then $P(A) \subset Q(A)$. On the other hand the mapping defined by $x_{12}(a) \mapsto y(a, \infty) \bmod P(A)$ and $x_{21}(A) \mapsto y(a, 0) \bmod P(A)$ respects relations (i) and (iii), hence yields a surjective homomorphism $\mathrm{ST}(2, A) \to Y(A)/P(A)$, so that we conclude $P(A) = Q(A)$. The subgroup $Q(A)$ contained in the free product $Y(A)$ is seen to be nonabelian. Since $\overline{K_2(2, A_k, A)} \simeq \pi_1 \mathrm{GL}(2, A)$ in virtue of Theorem 16, we have exhibited this fundamental group as an extension of $K_2(2, A)$ by a nonabelian group $Q(A)$.

We are at last in a position to compare the fundamental group of $\mathrm{GL}(n, A)$ with the Schur multiplier of its connected component $\mathrm{EL}(n, A)$. Combining Theorems 8, 16, 21 and the above, we obtain

THEOREM 22. *Let A be a semilocal unital algebra over a field k. Then $\pi_1 \mathrm{GL}(n, A) \simeq H_2(\mathrm{EL}(n, A), \mathbf{Z})$ for $n \geqslant 3$ unless $n = 3$ and $k = \mathbf{F}_2$, $\mathbf{F}_4$, or $n = 4$ and $k = \mathbf{F}_2$. Furthermore, $\pi_1 \mathrm{GL}(2, A)$ is an extension of $H_2(\mathrm{EL}(2, A), \mathbf{Z})$ by a nonabelian group $Q(A)$ when $k \neq \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5, \mathbf{F}_9$.*

This can be viewed as a stability result for the fundamental group, at least for commutative algebras. Indeed, for these semilocal algebras $\pi_1 \mathrm{GL}(n, A) \simeq H_2(\mathrm{EL}(n, A), \mathbf{Z}) \simeq K_2(n, A)$ remain stable for $n \geqslant 3$ (except possibly when $k = \mathbf{F}_3$), hence equal $\pi_1 \mathrm{GL}(A) \simeq H_2(\mathrm{EL}(A), \mathbf{Z}) \simeq K_2(A)$. This follows from [9; 12] where a presentation for $K_2(n, A)$ is announced which implies stability of these groups. A survey of known computations of $K_2(n, A)$ appears in [7]. For $n = 2$ the fit between fundamental group and Schur multiplier is less close.

It would be interesting to know what the situation is for other Chevalley groups. Though work on the Schur multiplier has been done [17–19], the methods of Dennis, Silvester, Cohn, and Bak featured in Section 5 have to my knowledge not been extended to other groups.

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